A UNITARY QUANTUM ELECTRODYNAMICS,

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Introduction

The Physical Society Conference held at the Cavendish Laboratory, Cambridge, in the summer of 1946, opened with three papers (1), presented by Pauli, Dirac, and Born, which were all devoted in some way to the elucidation of the difficulties of quantum electrodynamics. These difficulties had been known to exist since before 1930, but at first they seem to have been minimized as interesting curiosities which hardly detracted from the continually increasing successes of the quantum theory. It was only when the exploration of the territories newly opened up had passed from the pioneers to their numerous successors that the nature of the obstructing difficulties became the subject of renewed attention and concern. Naturally many suggestions for surmounting the difficulties were put forward, but these were mainly of a tentative nature which hardly commanded the confidence of the authors themselves. One of the earliest, which is especially relevant to the present paper, was made by Born and Rumer (2); reduced to its simplest terms, it was to introduce a factor e^{-k^2/b^2}, where b is a natural constant of the dimensions of a momentum, into the divergent integrals; but at that time this seemed a rather arbitrary procedure which therefore remained undeveloped.

Three general methods can be distinguished among the many suggestions for the removal of the difficulties. The first consists in the preliminary elimination of the difficulties already existing in the classical theory of the electron, with the object of subsequent quantization in accordance with the correspondence principle. An excellent summary of the progress accomplished in this direction, which includes also an extensive bibliography, has been published recently by Pais (3): he points out that two independent conditions have to be satisfied by any satisfactory relativistic theory of the electron - the self-energy must be finite, and the self-stress must be zero. Both conditions are satisfied by the theory of the Poincaré-electron, by Born's non-linear theory (4), and by the various two-field theories.

The second general method achieves the same object by effecting the reduction of the self-energy to zero; this is possible either by the use of special methods, such as the λ-limiting process, and the introduction of quanta with negative energy as suggested by Dirac (5), or by systematically discarding the divergent integrals as advocated by Heitler. These are the so-called 'subtraction' theories; a difficulty which has to be met by all of them has become apparent recently with the experimental discovery of the Lamb effect (6), which by common consent must be explained with the help of the disappearing terms.

At the Cambridge Conference it was pointed out by Pauli and Dirac that the quantum-mechanical perturbation theory, which is effectively a series expansion in powers of the fine-structure constant, introduces difficulties additional to those already apparent in classical electrodynamics. This is the starting point of the third generalmethod of approach, which consists in a critical examination of the perturbation theory. Peng (7), Feynmann (8), and Schwinger (9) have devised perturbation methods which appear to avoid some of the more obvious difficulties. The author (10) has attempted a rigorous solution of the perturbation problem, which, however, does not by itself liquidate the infinite transverse self-energy of the electron. It is, indeed, fairly obvious by now that what is lacking is not mathematical technique, but a fresh approach to electrodynamics, preferably by way of a new general principle.

In the third paper presented to the Cambridge Conference, Born gave an account of a general principle, called the Principle of Reciprocity, which he hoped would provide the key to the situation. The way in which this principle should be applied, however, depended on the correct physical identification of the quantities, called reciprocal invariants, which remain unaltered by four-dimensional Fourier transformation, and which, if the Principle of Reciprocity is correct, should play an outstanding role in the laws of nature; and at the time the physical significance of these reciprocal invariants was not clear. Soon afterwards, the author, engaged in an investigation of a general type of convergent field theory, discovered an operator - called in this paper the Lagrangian operator - which clearly played a fundamental role in physics, but was undetermined by any known physical law. Combining these separate aspects of the problem, Prof. Born and the author (11) were able to show that a coherent theory of the elementary particles emerged if it were supposed that the unknown operator should be a reciprocal invariant.

The object of the present paper is to examine the particular application of this theory to quantum electrodynamics. It is found that the adoption of the reciprocally invariant Lagrangian operator appropriate to the electromagnetic field removes all the difficulties of current quantum electrodynamics. In this respect alone it is clearly not unique; but the choice is not as wide as might be expected: a theory of the same type due to Podolsky (12), and suggested in another connection by Born (13), meets with difficulties which would be shared by a large number of similar theories. The specialization suggested by the Principle of Reciprocity avoids these difficulties and is, fundamentally, the simplest of all. Besides, it is not so much a question of knowing with certainty the right Lagrangian to adopt, as of finding <u>any</u> Lagrangian which does not lead to objectionable consequences. A positivist attitude is perhaps of assistance here. As long as one is subtracting infinities in the way which is necessary to secure results in agreement with experiments from the current theory, little confidence can be entertained concerning the correctness of the calculations; but <u>any</u> theory which is mathematically unobjectionable can be judged solely on whether its predictions are confirmed by experience. It is therefore very helpful to have at one's disposal a formalism which obtains the valid results of the current theory in a perfectly unquestionable way; for example, it will be possible to test the correctness of the various theories of the Lamb shift (14) by a straightforward, if perhaps rather tedious, application of the quantum formalism developed in the following pages.

Before embarking on a purely quantum-mechanical investigation, it seemed worth while to devote some attention to the corresponding classical problems. Here there was a precedent in the non-linear field theory, and, in spite of the fact that the fields determined by the Principle of Reciprocity are all linear, many similarities between the two theories could be detected. Both theories are unitary, in the sense that the electron can be regarded as a singularity in the electromagnetic field: a point at which the field equations, otherwise satisfied throughout the whole of space, are strictly violated. By means of this concept it is possible to treat the electromagnetic field and the electrons as an inseparable combination, thus realizing the idea of Mie, and the suggestion of J.J.Thomson that the rest-energy of the electron may be regarded as derived from the electromagnetic field. The latter concept was, of course, embodied in the old theories of Abraham, but discarded when it appeared to lead to relativistic difficulties: in fact, there are no relativistic difficulties when a point-singularity is considered.

A new problem arises when one proceeds to enquire after the equation of motion of the electronic singularity. In the non-linear field theory this was discussed by means of the principle of conservation of energy; Schrödinger (15), by postulating that the flux of energy and momentum across a small sphere surrounding the singularity must be balanced by changes associated with the singularity, was able to derive even the radiation reaction for Born's electron. A more sophisticated procedure, closer to recent developments in the orthodox theory, would be to derive the equation of motion from the variation of the Lagrangian, thus automatically ensuring the conservation of energy and momentum. Superficially this method appears rather strange in its application to a unitary theory, but the ever-present analogy with the orthodox theory guarantees a consistent interpretation. One point of Interest which emerges is that about two-thirds of the rest-energy associated with the singularity comes from what is _usually_ regarded as the field; this seems to be a general result not dependent on the particular field equations employed.

The quantization of the theory requires an advance into hitherto unexplored territory, as the method of Fock (16) adopted by Podolsky is clearly unsuited to present needs, and, owing to its non-linearity, it was never possible to quantize Born's theory of the electron in a satisfactory way. It might appear that the quantization of any unitary theory would meet with prohibitive difficulties in connection with the uncertainty principle. For the 'damping factor' in the interaction energy does not depend only on the coordinates of the electron - as assumed by McManus (17) in his recent paper on convergent theories - but also on the momentum, in a very intimate way. The relativistic invariant involved is not the four-dimensional distance, but the scalar of the four-dimensional angular momentum tensor. It is, moreover, patently impossible to follow the procedure of orthodox quantum mechanics (as presented, for example, by Schwinger) of replacing the current vector in Maxwell's equations by the quantity $\psi^{\dagger}\alpha_k\psi$ indicated by Dirac's theory; for this would mean that the charge is spread out in space when the position of the electron is indeterminate, so that the electrostatic self-energy depends on what quantities are assumed to be observable. This conflicts with the statistical interpretation of quantum mechanics, according to which not the charge, but the probability of finding it in a given position, is distributed throughout space when its position is indeterminate. Accordingly the electron remains a singularity in the photon field even in the quantum theory, but a probability amplitude for the position of the singularity has to be introduced. The result is that the wave function ψ of the electron, and the wave amplitude A of the photon field do not occur separately, but are merged into a single complex wave-function Φ representing the electrons and photons together in an inseparable union.

In spite of these fundamental modifications, it is possible to develop the theory of the interaction between a system of electrons and photons in a way which differs insensibly from the orthodox treatment. Dirac's equation for an electron in interaction with an electromagnetic field is only slightly modified, and the matrix elements of the interaction energy differ from those normally used only by a factor which appears innocent enough. It is not exactly the factor $e^{-k^2/2b^2}$ guessed by Born and Rumer in 1931, but its relativistic generalization $e^{\{(\underline{k}\cdot\underline{p})^2/(p^2+m^2c^2)-k^2\}/2b^2}$, where $\underline{p}$ is the momentum of the electron, in the ground state if the photon $\underline{k}$ is emitted, in the excited state if it is absorbed. This obviously provides an effective 'cut-off' for the transverse self-energy, and, indeed, for all the integrals associated with Heitler's 'round-about' transitions. On the other hand, it does not interfere with the real processes, even to the highest energies, except in a trifling and rather interesting way. This theory therefore provides some justification for the subtraction theories, but it shows also that they involve some small errors which, with the improvement of experimental technique, it may be possible to detect.

In the second appendix at the end of the paper, a section is included which indicates that it may be possible to develop a theory of the proton which is parallel to that of the electron. It is based on an original idea advanced by Born (18), and lately extended by K.C.Cheng, that the proton is simply a positive electron invested with an asymmetrical field of a magnetic character. This certainly accounts well enough for the mass of the proton, and there are indications that it can account for its other properties too.

1. Notation

Throughout this paper the convention of general relativity theory is applied to the suffixes k, l, m, n; the distinction between covariant and contravariant affixes is observed, and the summation of repreated affixes from 1 to 4 is understood, except where, for example, x_k occurs not as a factor, but standing for the four quantities x_1, x_2, x_3, x_4 in expressions such as $\rho(x_k, x_k')$. The metric tensor

$$g_{kl} = 0,\ k \neq l;\ = -1,\ k = l = 1, 2, 3;\ = 1,\ k = l = 4, \qquad (1.1)$$

of Galilean space-time is generally implied. Three-dimensional vectors are represented in Clarendon type (or underlined in manuscript). The coordinate four-vector is denoted by $x_k \equiv (\underline{x}, ct)$, and the momentum four-vector by $p_k \equiv (\underline{p}, \frac{E}{c})$. The symbol Ω denotes a volume of three-dimensional space which is selected for special consideration.

The suffixes α, β, γ are reserved to denote spin components, and the summation rule for repeated affixes applies also to these. Spin components will, however, be omitted where the practice can lead to no ambiguity. The letters sp, placed before an expression which is a spin matrix, signifies the trace, or sum of the diagonal elements, of the spin matrix.

The electronic charge e_0 and the four-vector potential of the electromagnetic field, together with derived field quantities, are measured in Heaviside units. For purposes of translation into ordinary units, a factor $(4\pi)^{\frac{1}{2}}$ has to be added to e_0, and a factor $(4\pi)^{-\frac{1}{2}}$ to all field variables.

2. Theory of Fields Without Interaction

The properties of fields, including those which are of special interest in quantum electrodynamics, are most conveniently derived from their Lagrangian densities. A procedure applicable to fields whose Lagrangian densities involve derivatives of the field variables not higher than the first has been reviewed by Pauli (19), and extended to fields with Lagrangian densities containing any number of derivatives of the field variables by Chang (20) and de Wet (21). It would be very cumbersome, if at all possible, however, to apply this procedure to the fields which are introduced later in this paper, and the method adopted is therefore a development of one introduced by the author in some previous publications (10, 11), to which it may be necessary to refer.

In the present exposition the theory of real fields is considered first, because it can be shown that, in contrast to the usual theory, the theory of complex fields follows simply from that of real fields. First, however, it is necessary to examine briefly the properties of the operator ζ whose coordinate representative is

$$\zeta(x_k, x_k') = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \delta(x_k - x_k' - \epsilon_k \tau) \frac{d\tau}{\tau}, \qquad (2.1)$$

where P denotes the principal value of the integral, and ϵ_k is a small time-like four-vector, such that

$$\epsilon_k \epsilon^k > 0 , \quad \epsilon_4 > 0 . \qquad (2.2)$$

In a special Lorentz frame, ϵ_k may be taken to be $(0,0,0,\epsilon)$, $\epsilon > 0$. The operator ζ, though as far as the author is aware it does not occur explicitly in the literature, is implicit in much work in quantum electrodynamics, notably the recent work of Schwinger (9). It can be expressed as the difference $e_1 - e_2$, where

$$\left.\begin{aligned} e_1(x_k, x_k') &= \frac{1}{2\pi i} \int_+ \delta(x_k - x_k' - \epsilon_k \tau) \frac{d\tau}{\tau}, \\ e_2(x_k, x_k') &= -\frac{1}{2\pi i} \int_- \delta(x_k - x_k' - \epsilon_k \tau) \frac{d\tau}{\tau}, \end{aligned}\right\} \qquad (2.3)$$

and both integrals are from $-\infty$ to $+\infty$, the first passing below the origin and the second above the origin in the complex τ-plane. It is readily shown that e_1, operating on a state vector ψ, annihilates the negative frequencies, and e_2 the positive frequencies in the harmonic analysis with respect to time, thus:

$$\left.\begin{aligned} e_1\psi &= \psi^+ , & e_2\psi &= \psi^-, \\ \psi e_1 &= \psi^- , & \psi e_2 &= \psi^+. \end{aligned}\right\} \qquad (2.4)$$

For if $\psi = e^{-icp_4 t/\hbar}$, $e_1\psi = \frac{1}{2\pi i}\int_+ e^{-icp_4(t-\epsilon\tau)} \frac{d\tau}{\tau}$, which vanishes if $p_4 < 0$, as can be seen by completing the contour with an infinite semi-circle in the lower half-plane, and reduces to $e^{-icp_4t/\hbar}$ if $p_4 > 0$, as can be seen by completing the contour with an infinite semi-circle in the upper half-plane. This is sufficient to establish the first of the relations (2.4), and the others are similar. The following are simple consequences:

$$\left.\begin{array}{l} e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1e_2 = e_2e_1 = 0, \\ e_1 + e_2 = 1, \quad e_1 = \tfrac{1}{2}(1+\zeta), \quad e_2 = \tfrac{1}{2}(1-\zeta), \\ \zeta^2 = 1, \quad e_1\zeta = e_1, \quad e_2\zeta = -e_2. \end{array}\right\} \qquad (2.5)$$

Now let $\rho_{\alpha\beta}(x_k, x_k')$ represent the relativistic statistical matrix in the coordinate representation, a real function of the two sets of coordinate variables $x_k \equiv (\underline{x}, ct)$ and $x_k' \equiv (\underline{x}', ct')$, and with spin affixes α, β. This may generally be factorized, thus:

$$\rho_{\alpha\beta}(x_k, x_k') = \sigma_{\alpha\gamma}(x_k)\sigma_{\gamma\beta}(x_k'), \qquad (2.6)$$

where $\sigma_{\alpha\beta}(x_k)$ is the corresponding w ave function. The wave operator $F_{\alpha\beta}(p_k, p_k')$ is defined in such a way that the representative of what may be called the <u>Lagrangian operator</u>,

$$\left.\begin{array}{l} \mathcal{L}(x_k, x_k') = F_{\alpha\beta}(p_k, p_k')\rho_{\beta\alpha}(x_k, x_k') = \mathrm{sp}\, F\rho, \\ p_k = i\hbar\dfrac{\partial}{\partial x^k}, \quad p_k' = -i\hbar\dfrac{\partial}{\partial x^{k'}}, \end{array}\right\} \qquad (2.7)$$

reduces to the Lagrangian density when x_k is set equal to x_k'. It has been shown by the author in the papers referred to that the field equations derived from this Lagrangian have the form

$$F_{\alpha\beta}(p_k, p_k)\sigma_{\beta\gamma}(x_k) = 0 \quad , \quad \left(p_k = i\hbar\frac{\partial}{\partial x^k}\right), \tag{2.8}$$

and that the properties of the field are very readily described in terms of the operator $\mathcal{G}^k_{\alpha\beta}(p_k, p_k')$ defined by

$$\left.\begin{aligned} F_{\alpha\beta}(p_k', p_k') &= F_{\alpha\beta}(p_k, p_k') + (p_k' - p_k)\,\mathcal{G}^k_{\alpha\beta}(p_k, p_k'), \\ F_{\alpha\beta}(p_k, p_k) &= F_{\alpha\beta}(p_k, p_k') + (p_k - p_k')\,\mathcal{G}^{k\prime}_{\alpha\beta}(p_k, p_k'). \end{aligned}\right\} \tag{2.9}$$

The four-vector

$$R^k(x_k) = sp\left\{\tfrac{1}{2}(\mathcal{G}^k + \mathcal{G}^{k\prime})\rho\right\}(x_k, x_k), \tag{2.10}$$

which was interpreted as the charge-current vector in the earlier paper, vanishes, as it should, for a real field, but the density-flux density vector, defined by

$$D^k(x_k) = sp\left\{\tfrac{1}{2}(\mathcal{G}^k + \mathcal{G}^{k\prime})\zeta\rho\right\}(x_k, x_k) \tag{2.11}$$

does not vanish. This shows that the operator ζ defined in (2.1) is essentially a charge annihilation operator. The four-vector D^k is readily shown to satisfy a conservation equation by multiplying the equations (2.9) by $\zeta\rho$, and subtracting one from another. With the help of (2.8), this leads to

$$\left(\frac{\partial}{\partial x^k} + \frac{\partial}{\partial x^{k\prime}}\right)\left\{(\mathcal{G}^k + \mathcal{G}^{k\prime})\zeta\rho\right\}(x_k, x_k') = 0\ ; \quad \frac{\partial D^k}{\partial x^k} = 0. \tag{2.12}$$

Similarly, by multiplying the equations (2.9) by

$\frac{1}{2}(p^{l}+p^{l\prime})\rho$ before subtraction, it is possible to show that

$$\mathcal{P}^{kl}(x_k) = \mathrm{sp}\left\{\tfrac{1}{4}(\mathcal{G}^{k}+\mathcal{G}^{k\prime})(p^{l}+p^{l\prime})\rho\right\}(x_k,x_k) \qquad (2.13)$$

satisfies the conservation equation $\frac{\partial \mathcal{P}^{kl}}{\partial x^k}=0$. The tensor $\mathcal{P}^{kl}$ will be identified with the true energy-momentum tensor; it has been shown before that it differs from the canonical tensor

$$\mathcal{L}^{kl}(x_k) = \mathrm{sp}\left\{\tfrac{1}{2}(\mathcal{G}^{k}p^{l}+\mathcal{G}^{k\prime}p^{l\prime}-Fg^{kl})\rho\right\}(x_k,x_k) \qquad (2.14)$$

only by a tensor which vanishes on integration over the real space Ω. The symmetrical energy-momentum tensor, which is closely associated with the angular momentum tensor, will not be required in this work, but as the derivation for a density matrix with spin differs in detail from that given before, it is included in the first appendix.

The commutation rules for the field may be stated in the form

$$\tfrac{1}{2}(\mathcal{G}^{4}+\mathcal{G}^{4\prime})\{(\zeta\rho)(x_k,x_k')-(\rho\zeta)(x_k',x_k)\} = i\hbar\,\delta(\underline{x}-\underline{x}'), \quad t'=t. \qquad (2.15)$$

This form may be obtained from the corresponding Lorentz-invariant statement

$$\int \tfrac{1}{2}(\mathcal{G}^{k}+\mathcal{G}^{k\prime})(\zeta\rho-\rho\zeta)\,dS_k = i\hbar \qquad (2.16)$$

where S is any surface of three dimensions with space-like connection, by first observing that on account of (2.12), (2.16) holds independently of the particular surface S chosen, and then choosing the surface $t =$ const. The connection of (2.15) with the usual commutation rule for real fields may be exhibited by re-writing it in the form

$$\tfrac{1}{2}(\mathcal{G}^{4}+\mathcal{G}^{4\prime})\{\sigma^{+}(x_k)\sigma^{-}(x_k')-\sigma^{-}(x_k')\sigma^{+}(x_k)\}=i\hbar\,\delta(\underline{x}-\underline{x}'),\ t'=t, \tag{2.17}$$

with the help of (2.4) and (2.5).

It can now be seen how the theory of a complex field may be derived simply from that of the real field just considered. It is necessary only to multiply ρ by e_1 before and after to obtain the complex statistical matrix $\sigma^{+}_{\alpha\gamma}(x_k)\sigma^{-}_{\gamma\beta}(x_k')$, which can be substituted everywhere (except in the commutation rule) for ρ. Since $\zeta e_1 = e_1$, the charge-current vector does not then vanish, but becomes identical with the density-flux density vector (2.11). The commutation rule (2.15) cannot, however, be retained, as the left-hand side vanishes for $x_k' = x_k$; it must be replaced by

$$\tfrac{1}{2}(\mathcal{G}^{4}+\mathcal{G}^{4\prime})\{(\zeta\rho)(x_k,x_k')+(\rho\zeta)(x_k',x_k)\}=i\hbar\,\delta(\underline{x}-\underline{x}'),\ t'=t, \tag{2.18}$$

which indicates that the particle now satisfies Fermi statistics instead of Bose statistics.

This, however, does not represent the only possibility; one may replace ρ by $e_2\rho e_2$ instead of by $e_1\rho e_1$. As $\zeta e_2 = -e_2$, the charge is now opposite in sign to the density, which is formally negative: real particles must be regarded as <u>holes</u>

in a distribution present, though unobservable, *in vacuo*.

In this way, the theory of real and complex fields is readily obtained from the same formalism. The application of this theory to the problems of electrodynamics will be examined in subsequent sections.

3. Application to the Pure Radiation Field

The Lagrangian functions allowed by the theory of the preceding section, and expressed by the equation (2.7), are clearly very arbitrary, even when the spin and rest-masses of the particles contained in the field are specified. For example, the wave operator of the pure radiation field can have the form

$$F^{k}_{\cdot l}(p_k, p_k) = f(p^k p_k)(p^m p_m \delta^{k}_{l} - p^k p_l) \qquad (3.1)$$

where the function f is quite arbitrary. The choice of ordinary electrodynamics corresponds to $f = -\hbar^{-2}$. Podolsky (12) has suggested, effectively, that one should take instead

$$f = -\hbar^{-2}(1 - p_k p^k / b^2)$$

where b is a large quantity of the dimensions of a momentum, and has shown that this, together with some other formalism, leads to a finite electrodynamics; unfortunately this particular choice also leads to some strange consequences, among them the existence in the electromagnetic field ofparticles of mass b/c. These might very well be supposed to be mesons, but for the fact that they would be necessarily associated with intense fields in the neighbourhood of the electron, without independent existence. Clearly if one wishes to avoid unphysical consequences of this sort, the equation $f(m^2c^2) = 0$ must have no real roots; but this condition alone does not suffice to specify the function, and for proper guidance a general principle is obviously required.

Some time ago it was suggested to the author by Prof. M. Born that the Principle of Reciprocity of which he and others (22) had sought to make use without important result in other connections many years earlier might provide the key to this problem, through the postulate that for all particles the function $F(p_k, p_k)$ should be a reciprocal invariant: a function which is unchanged by four-dimensional Fourier transformation, and thus preserves the fundamental symmetry between coordinates and momenta already apparent in classical and quantum mechanics. The fact that this principle proved to be fruitful in predicting the rest-masses, in sufficient agreement with the experimental values, of the newly discovered π- and μ-mesons added considerably to its inherent plausibility. Further, it had been shown many years ago by Born and Rumer (2) that a simple way of overcoming at least some of the divergence difficulties of quantum electrodynamics is to introduce a factor $e^{p_k p^k/b^2}$ of the form predicted by the Principle of Reciprocity into the divergent integrals, and <u>any</u> general principle which eliminates the divergence difficulties is worthy of consideration on this ground alone.

When, however, the Principle of Reciprocity was applied to the problems of quantum electrodynamics in the rather crude way mentioned above, two objections soon became apparent. The first was that the self-reciprocal wave operator

$$F^{k}_{\cdot l}(p_k, p_k) = -\hbar^{-2} e^{-p_k p^k/2b^2}(p^m p_m \delta^k_l - 4p^k p_l)$$

suggested in this way by the theory of reciprocity, is not of the form (3.1). A much more serious objection was that this application of the Principle did not show how the function $F^{k}_{\cdot l}(p_k, p_k')$ was to be obtained from $F^{k}_{\cdot l}(p_k, p_k)$, a question which is unimportant for pure fields, but essential to the theory of fields in interaction. This question has been answered by a proposal made in the first instance by Prof. Born, and elaborated by Dr. K.C.Cheng, that $F^{k}_{\cdot l}(p_k, p_k')$ should be assumed to be reciprocally invariant under Fourier transformation with respect to both p_k and p_k'. The mass-spectrum resulting from this postulate is the same as before, and for the special instance of the electromagnetic field one has unambiguously

$$\left.\begin{aligned} f(p_k p^k) &= -\hbar^{-2} e^{-p^k p_k/b^2} \\ F^{k}_{\cdot l}(p_k, p'_k) &= -\hbar^{-2} e^{-(p_k p^k + p'_k p'^k)/2b^2}(p'_m p^m \delta^k_l - p'_l p^k) \end{aligned}\right\} \qquad (3.2)$$

If this is accepted, the form adopted by Podolsky is evidently a first approximation to the correct function, from which it is obtainable by expanding the exponential function in series.

The statistical matrix for the radiation field will first be taken in the usual form

$$\rho^{k}_{\cdot l}(x_k, x'_k) = \tfrac{1}{2} A^k(x_k) A_l(x'_k), \qquad (3.3)$$

and it will be supposed that a gauge transformation has been effected, such that

$$\frac{\partial A^k}{\partial x^k} = 0 \qquad (3.4)$$

identically, and not, as is sometimes assumed, as an expectation value: Fermi's procedure is not, in the author's opinion, wholly satisfactory. The field equation (2.8) then reduces to

$$e^{a^2\Box}\,\Box A_k = 0 \quad , \quad a = \hbar/b, \qquad (3.5)$$

and is satisfied by

$$A_k(x_k) = \Omega^{-1} \sum_{\mathbf{p}} \{ A_k(\mathbf{p}) e^{-ip_k x^k/\hbar} + A^*_k(\mathbf{p}) e^{ip_k x^k/\hbar} \} \qquad (3.6)$$

provided $p_4 = p = |\mathbf{p}|$. The auxiliary condition (3.4) requires that $p_k A^k(\mathbf{p}) = 0$, and as a four-vector perpendicular to a null-vector has only two independent components, one can write

$$\left.\begin{aligned} A_k(\mathbf{p}) &= A_k^{(1)}(\mathbf{p}) + A_k^{(2)}(\mathbf{p}), \\ A_k^{(1)}(\mathbf{p}) = A^{(1)}(\mathbf{p})n_k^{(1)}, &\quad A_k^{(2)}(\mathbf{p}) = A^{(2)}(\mathbf{p})n_k^{(2)}, \\ n_k^{(1)}n^{(1)k} = n_k^{(2)}n^{(2)k} = 1, &\quad n_k^{(1)}n^{(2)k} = 0, \end{aligned}\right\} \qquad (3.7)$$

so that $n_k^{(1)}$ and $n_k^{(2)}$ are two <u>orthogonal unit</u> vectors perpendicular to p_k.

From (2.9) one has

$$\begin{aligned} (p_m' - p_m)\mathcal{G}_l^{km}(p_k, p_k') = -\hbar^{-2}(p_m' - p_m)e^{-p_k p^{k\prime}/2b^2}\Big[(p^{m\prime} + p^m)\times \\ \times \frac{e^{-p_k' p^{k\prime}/2b^2} - e^{-p_k p^k/2b^2}}{p_k' p^{k\prime} - p_k p^k}(p_n' p^{n\prime}\delta_l^k - p^{k\prime}p_l') + e^{-p_k p^k/2b^2}(p^{m\prime}\delta_l^k - g^{mk}p_l')\Big], \end{aligned} \qquad (3.8)$$

$$\mathcal{G}_{\cdot k}^{\,\cdot lm\prime}(p_k, p_k') = \mathcal{G}_{\cdot k}^{lm}(p_k', p_k).$$

Hence, by direct substitution in (2.11), and making use of (3.4) and (3.5),

$$D_k(x_k) = \frac{i}{4\hbar}\{(\zeta B^l)(x_k)E_{kl}(x_k) - (\zeta E_{kl})(x_k)B^l(x_k)\}, \qquad (3.9)$$

where

$$B_k(x_k) = e^{-p_k p^k/2b^2}A_k(x_k), \quad E_{kl} = \frac{\partial B_l}{\partial x^k} - \frac{\partial B_k}{\partial x^l}. \qquad (3.10)$$

Similarly, by substitution into (2.14),

$$\mathcal{L}_{kl}(x_k) = \frac{1}{4}\Big\{\Big(\frac{\partial B^m}{\partial x^l}E_{km} + E_{km}\frac{\partial B^m}{\partial x^l}\Big) - \Big(\frac{\partial B_m}{\partial x^n}E^{nm} + E^{nm}\frac{\partial B_m}{\partial x^n}\Big)g_{kl}\Big\}, \qquad (3.11)$$

and in this notation the Lagrangian operator of (2.7) becomes

$$\mathcal{L}(x_k, x_k') = \frac{1}{4}E_{mn}(x_k)E^{nm}(x_k'). \qquad (3.12)$$

If (3.6) is substituted into (3.9), and the result integrated over all space, one has, for the total 'amount of radiation' (number of photons)

$$\begin{aligned} N = \int D_4(x_k)\,d\Omega = \frac{\Omega^{-2}}{4\hbar^2}\int\sum_{\mathbf{p},\mathbf{p}',i}\Big[-p_4'\{A^{(i)}(\mathbf{p})e^{-ip_kx^k/\hbar} - A^{(i)*}(\mathbf{p})e^{ip_kx^k/\hbar}\}\{A^{(i)}(\mathbf{p}')e^{-ip_k'x^k/\hbar} - A^{(i)*}(\mathbf{p}')e^{ip_kx^k/\hbar}\} \\ + p_4\{A^{(i)}(\mathbf{p})e^{-ip_kx^k/\hbar} + A^{(i)*}(\mathbf{p})e^{ip_kx^k/\hbar}\}\{A^{(i)}(\mathbf{p}')e^{-ip_k'x^k/\hbar} + A^{(i)*}(\mathbf{p}')e^{ip_k'x^k/\hbar}\}\Big]d\Omega \\ = \frac{\Omega^{-1}}{2\hbar^2}\sum_{\mathbf{p},i}p_4\{A^{(i)}(\mathbf{p})A^{(i)*}(\mathbf{p}) + A^{(i)*}(\mathbf{p})A^{(i)}(\mathbf{p})\}, \end{aligned} \qquad (3.13)$$

where i, as indicated in (3.7), takes only the values 1 and 2; and similarly from (3.11) one obtains the total energy and momentum in the form

$$P_k = \int L_{4k}(x_k)\,d\Omega = \frac{\Omega^{-1}}{2\hbar^2}\sum_{\mathbf{p},i} p_4 p_k \{A^{(i)}(\mathbf{p})A^{(i)*}(\mathbf{p}) + A^{(i)*}(\mathbf{p})A^{(i)}(\mathbf{p})\}. \qquad (3.14)$$

The commutation rules, obtained from (2.16) or (2.17), are

$$\frac{\Omega^{-1}}{\hbar^2} p_4 [A^{(i)}(\mathbf{p}), A^{(j)*}(\mathbf{p}')] = \delta_{\mathbf{p}\mathbf{p}'}\,\delta_{ij}\,; \qquad (3.15)$$

they have the effect of ensuring that the quantities

$$N^{(i)}(\mathbf{p}) = \frac{\Omega^{-1}}{2\hbar^2} p_4 \{A^{(i)}(\mathbf{p})A^{(i)*}(\mathbf{p}) + A^{(i)*}(\mathbf{p})A^{(i)}(\mathbf{p})\} \qquad (3.16)$$

which appear in (3.13) and (3.14) are (apart from an inconvenient residue of one half) integers, and the particle interpretation of the field follows in the usual way.

It is by now obvious that the introduction of the factor $e^{-(p_k p^k + p'_k p'^k)/2b^2}$ into the Lagrangian has no observable consequences in the theory of the pure radiation field; indeed, by regarding B_k and E_{kl} as the electromagnetic vector potential and field tensor, it is reduced to a form indistinguishable from the usual theory. As will appear later, however, the situation is quite different when a unitary standpoint is adopted, and point singularities are supposed to exist in the field simulating the behaviour of electrons and positrons. From one point of view, every electromagnetic field which does not vanish everywhere may be attributed to such singularities, as even the pure radiation field can be regarded as generated by a distribution of singularities at infinity.

It has been noticed above, and is indeed very well known, that the $N^{(i)}$'s are not truly integers, but half-odd integers. This leads to an infinite 'zero-point' energy $\Sigma\frac{1}{2}h\nu$ for the radiation field which is fortunately the least embarrassing of the divergent terms, as it can be eliminated in a Lorentz-invariant way by a variety of means, of which the most satisfactory is probably an emendment of the correspondence principle to the effect that products of A_k and A_l^* shall be written in the order $A_l^* A_k$ in passing from the classical to the quantum theory. This is assured automatically if, instead of the statistical matrix (3.3), one adopts $\rho = (A_k e_1)(x_k)(e_1 A_l)(x_l)$, which obviously leads to the same observable consequences, without the awkward zero-point energy.

For future convenience the preceding theory of the radiation field will be translated into a slightly different form, embodying this improvement. Let α_k represent Dirac's spin operators, satisfying the relation

$$\alpha_k \alpha_l + \alpha_l \alpha_k = 2g_{kl}, \qquad (3.17)$$

so that although α_4 is hermitian, $\underline{\alpha}$ is anti-hermitian. If

$$\eta = \alpha^k p_k \quad , \quad \eta' = (\alpha^k p'_k)^* \tag{3.18}$$

then the operator $F(\eta, \eta')$ defined by

$$F(\eta, \eta') = -\hbar^{-2} \eta' \eta \, e^{-(\eta^2 + \eta'^2)/2b^2} \tag{3.19}$$

is clearly self-reciprocal in the sense already explained. If now the statistical operator is expressed in the form

$$\rho = A(x_k) \rho_0 A^*(x'_k) \quad , \quad A(x_k) = \alpha^k A_k(x_k) \tag{3.20}$$

where $\rho_0 = \phi\phi^*$ is a normalized spin operator not involving x_k , x'_k , or $\alpha_5 = \alpha_1\alpha_2\alpha_3\alpha_4$, then the ~~Lagrangian~~ Lagrangian operator is

$$\mathcal{L}(x_k, x'_k) = -\hbar^{-2} \, \mathrm{sp} \{ \eta e^{-\eta^2/2b^2} \rho \, \eta' e^{-\eta'^2/2b^2} \}. \tag{3.21}$$

It is easy to verify that this reduces to a form equivalent to (3.12), in the following way:

$$\begin{aligned}
\eta e^{-\eta^2/2b^2} A(x_k) &= \{ \tfrac{1}{2}(\alpha^k\alpha^l + \alpha^l\alpha^k) + \tfrac{1}{2}(\alpha^k\alpha^l - \alpha^l\alpha^k) \} p_k B_l(x_k) \\
&= \tfrac{1}{2} i\hbar \left(\frac{\partial B_k}{\partial x_k} + \alpha^k\alpha^l E_{kl} \right) = \tfrac{1}{2} i\hbar \, \alpha^k\alpha^l E_{kl} \\
A^*(x'_k) \eta' e^{-\eta'^2/2b^2} &= -\tfrac{1}{2} i\hbar \{ \alpha^k\alpha^l E_{kl}(x'_k) \}^* \\
\mathrm{sp}\{ \eta e^{-\eta^2/2b^2} \rho \, \eta' e^{-\eta'^2/2b^2} \} &= \tfrac{1}{4}\hbar^2 \phi^* \{ E^*_{kl}(x'_k) E^{kl}(x_k) + i\alpha_5 \epsilon^{klmn} E^*_{kl}(x'_k) E_{mn}(x_k) \} \phi
\end{aligned} \tag{3.22}$$

As ρ_0 has been defined so that $\mathrm{sp}\,\rho_0 = \phi^*\phi = 1$ and $\mathrm{sp}\,\alpha_5\rho_0 = \phi^*\alpha_5\phi = 0$, the equivalence of the two forms is apparent. The field equations derived from (3.21) are simply

$$e^{-\eta^2/b^2} \eta^2 A \rho_0 = 0 \tag{3.23}$$

One has, further,

$$\begin{aligned}
(p'_k - p_k) \mathcal{G}^k(\eta, \eta') &= (p'_k - p_k) e^{-\eta'^2/2b^2} \left\{ (p^{k\prime} + p^k) \frac{e^{-\eta'^2/2b^2} - e^{-\eta^2/2b^2}}{\eta'^2 - \eta^2} + e^{-\eta^2/2b^2} \eta' \alpha^k \right\}, \\
\mathcal{G}^{k\prime}(\eta, \eta') &= e^{-\eta^2/2b^2} \left\{ (p^k + p^{k\prime}) \frac{e^{-\eta^2/2b^2} - e^{-\eta'^2/2b^2}}{\eta^2 - \eta'^2} + e^{-\eta'^2/2b^2} \alpha^{k*} \eta \right\}
\end{aligned} \tag{3.24}$$

Hence, from (2.11), (2.13) and (2.14),

$$\begin{aligned} \mathcal{D}^k &= \phi^* B^*(x_k)(\eta'\alpha^k + \alpha^{k*}\eta)\, B(x_k)\phi \,, \quad B = e^{-\eta^2/2b^2} A \,, \\ p^{kl} &= \phi^* B^*(x_k) \cdot \tfrac{1}{2}(\eta'\alpha^k + \alpha^{k*}\eta)(p^l + p^{l\prime}) \cdot B(x_k)\, \phi \\ \mathcal{L}^{kl} &= \phi^* B^*(x_k)(\eta'\alpha^k p^l + \alpha^{k*} p^{l\prime}\eta - \eta'\eta)\, B(x_k)\, \phi \end{aligned} \tag{3.25}$$

In this formalism, the field theory is somewhat simpler than before, as well as being better adapted to the quantum electrodynamics to be developed later in this paper.

4. Singularities in the Radiation Field

The pure radiation field, whose properties were examined in the previous section, is seen on careful consideration to be an abstraction far removed from physical reality. With the help of the commutation rules it can be analysed into photons, a definite number for each state of momentum and energy. The number of photons with an assigned momentum is necessarily the same for all time: in technical language, a constant of the motion. It is obvious without a very deep inquiry into the philosophy of measurement that such an assembly would be, in principle, unobservable: a particle of any kind can be detected only by allowing it to interact in some way with other particles. The conclusion is inescapable that it is physically meaningless to consider an assembly of photons apart from the electrons and positrons, through the existence of which they become susceptible to observation. It would, however, be equally meaningless to suppose that electrons and positrons could exist, in the sense of being physically observable, independently of the radiation field. The field and charged particles together form a complex system whose mutual relations alone may be supposed to constitute the elements of experience. The investigation of the nature of these relations is evidently the fundamental task of electrodynamics.

The solution of this problem attempted in the present paper is by a method originally suggested by Abraham (23), but to some extent discredited when it was found to lead to difficulties in connection with relativistic invariance. That these difficulties could not be genuinely relativistic in origin, however, became apparent when, Einstein, the principal architect of the theory of relativity, solved the analogous problem of the relation between inertia and the gravitational field by the same <u>kind</u> of method. This was confirmed with the development by Born of a unitary non-linear electrodynamics, the only disadvantage of which was the analytical difficulty which impeded the quantization of the non-linear field. The common idea of these theories was that the particle should appear as a singularity in its associated field, particle and field thus constituting an inseparable whole. The present object is to help to carry the unitary programme towards its ultimate success.

The determination of a function $Y(\underline{x})$ which makes $\frac{e_0}{a} Y(\underline{x}) \delta_k^4$ a solution of the field equations (3.5) everywhere except at the origin, i.e., such that

$$e^{-\eta^2/b^2} \eta^2 Y(\underline{x}) = -a\hbar^2 \delta(\underline{x}) \tag{4.1}$$

will first be attempted. It may obviously be assumed that $Y(\underline{x})$ is independent of time. By introducing the Fourier transformation

$$Y(\underline{x}) = \frac{1}{(2\pi\hbar)^3} \int \overline{Y}(\underline{k}) e^{i\underline{k}\cdot\underline{x}/\hbar} d\underline{k} \tag{4.2}$$

one obtains from (4.1)

$$e^{k^2/b^2} k^2 \overline{Y}(\underline{k}) = a\hbar^2, \quad \overline{Y}(\underline{k}) = a\hbar^2 k^{-2} e^{-k^2/b^2} \tag{4.3}$$

and hence

$$\begin{aligned} Y(\underline{x}) &= \frac{a\hbar^2}{(2\pi\hbar)^3} \int_0^\infty \int_0^\pi k^{-2} e^{-k^2/b^2} e^{ikx\cos\theta/\hbar} 2\pi \sin\theta \, d\theta \, k^2 dk \\ &= \frac{a}{2\pi^2 x} \int_0^\infty k^{-1} e^{-k^2/b^2} \sin(kx/\hbar) \, dk \\ &= \frac{a}{(2\pi)^{3/2}\sqrt{2}\,x} \int_0^{x/a} e^{-\frac{1}{4}u^2} du . \end{aligned} \tag{4.4}$$

This is the field due to a stationary singularity at the origin. For $x \gg a$, and so at all measurable distances from the electron, the field $\frac{e_0}{a} Y(x)$ is substantially the Coulomb field of a charge e_0 (which is $e_0/4\pi x$ in Heaviside units), but is finite everywhere, and has the finite total energy

$$m_0 c^2 = \frac{1}{2} \int \frac{e_0}{a} Y(x) e_0 \delta(\underline{x}) d\Omega = \frac{e_0^2}{2a} Y(0) = \frac{e_0^2}{(4\pi)^{3/2} a} \tag{4.5}$$

It will be seen later that this is really about one third of the rest-energy associated with a charge e_0 , as other important contributions come from the interaction with the radiation field. This serious discrepancy is due simply to the fallacy of supposing that the electron can exist independently of the radiation field.

The solution just obtained for the field of a point charge has besides two obvious defects: it is non-relativistic, being applicable only to a charge at rest, and contradicts Heisenberg's uncertainty principle, according to which it would be impossible to specify exactly the position of a stationary charge, or one whose velocity is known. The latter question will be considered later in the sections devoted exclusively to the quantum theory. The relativistic difficulty is not, however, difficult to overcome: by making a Lorentz transformation of axes to a system moving with velocity $-\underline{v}$ relative to that previously considered,

$$\underline{x} \longrightarrow \tilde{\underline{x}} = \{ \underline{v} \wedge (\underline{x} \wedge \underline{v}) + \gamma(\underline{x} \cdot \underline{v} - v^2 t)\underline{v} \}/v^2, \qquad \gamma = (1 - v^2/c^2)^{-\frac{1}{2}} \tag{4.6}$$

and using the fact that the original equation is Lorentz-invariant, one obtains immediately the corresponding solution

$$A_k = \frac{e_0}{a}\frac{\gamma v_k}{c} Y(\tilde{x}) \quad , \qquad v_k = (\underline{v}, c) \tag{4.7}$$

The scalar invariant $\tilde{x} = |\tilde{\underline{x}}|$ can be expressed in terms of the angular momentum of the charge about an origin in space-time. Defining the angular momentum tensor by

$$m_{kl} = x_k p_l^0 - x_l p_k^0 \quad , \qquad p_k^0 = m\gamma v_k \, , \tag{4.8}$$

so that

$$\mu^2 = \tfrac{1}{2} m_{kl} m^{lk} = \underline{n}^2 - \underline{m}^2, \ \underline{n} = mc\gamma(\underline{x} - \underline{v}t), \ \underline{m} = m\gamma \underline{x} \wedge \underline{v} \tag{4.9}$$

one obtains

$$\begin{aligned} \mu^2 &= m^2\gamma^2\{x^2c^2 - 2\underline{x}\cdot\underline{v}c^2t + v^2c^2t^2 - x^2v^2 + (\underline{x}\cdot\underline{v})^2\} \\ &= \frac{m^2\gamma^2}{v^2/c^2}\left[(\underline{x}\cdot\underline{v} - v^2t)^2 + (1 - v^2/c^2)\{x^2v^2 - (\underline{x}\cdot\underline{v})^2\}\right] \\ &= m^2c^2\tilde{x}^2 \end{aligned} \tag{4.10}$$

The invariant $\tilde{x}$ is also closely related to the retarded distance of the point-instant x_k from the path of the singularity, which is, in fact, easily seen to be $\gamma^{-1}\tilde{x}$ with $t = x/c$.

To form the Lagrangian from the potential $\frac{e_0}{a}\frac{\gamma v_k}{c} Y\left(\frac{\mu}{mc}\right)$, it is necessary to evaluate also the 'pseudo-potential'

$$B_k = \frac{\cancel{e_0\gamma v_k}}{ac} = \frac{e_0\gamma v_k}{ac} e^{-\eta^2/2b^2} Y\left(\frac{\mu}{mc}\right) \tag{4.11}$$

for the field of the singularity. When $\underline{v} = 0$, one has, according to (4.4),

$$\begin{aligned} Z(x) &= e^{k^2/2b^2} Y(x) = \frac{a}{2\pi^2 x}\int_0^\infty k^{-1} e^{-k^4/2b^2} \sin(kx/\hbar)\, dk \\ &= \frac{a}{(2\pi)^{3/2} x}\int_0^{x/a} e^{-\frac{1}{2}u^2}\, du \end{aligned} \tag{4.12}$$

The Fourier transform of the function $Z\left(\frac{\mu}{mc}\right)$ will often be required; this is most easily obtained as follows. If $\underline{v} = 0$, the <u>four-dimensional</u> Fourier transform of $Z(x)$ is

$$\int Z(x)\, e^{-i\underline{k}\cdot\underline{x}/\hbar}\, d\Omega \int e^{ik_4x_4/\hbar}\, dx_4 = a\hbar^2 k^{-2} e^{-k^2/2b^2} \cdot 2\pi\hbar\, \delta(k_4),$$

of which the relativistic generalization is clearly

$$a\hbar^2 (-k_\ell k^\ell)^{-1} e^{k_\ell k^\ell/2b^2} \cdot 2\pi\hbar\, \delta(p_\ell k^\ell/mc)$$

Hence

$$\int Z\left(\frac{\mu}{mc}\right) e^{-i\underline{k}\cdot\underline{x}/\hbar}\, d\Omega = a\hbar^2 \int \frac{e^{(k_4^2-\underline{k}^2)/2b^2}\, mc\, \delta(p_4k_4 - \underline{p}\cdot\underline{k})\, e^{-ik_4x_4/\hbar}\, dk_4}{\underline{k}^2 - k_4^2}$$

$$= \frac{a\hbar^2\gamma^{-1} e^{-i\underline{v}\cdot\underline{k}t/\hbar}\, e^{\{(\underline{k}\cdot\underline{v})^2/c^2 - k^2\}/2b^2}}{k^2 - (\underline{k}\cdot\underline{v})^2/c^2} \qquad (4.13)$$

Now the Lagrangian of the field of the singularity, according to (3.12), is

$$L^e = \frac{1}{4}\int \frac{e_0^2\gamma^2}{a^2}\left(v_k \frac{\partial Z}{\partial x_\ell} - v_\ell \frac{\partial Z}{\partial x_k}\right)\left(v^\ell \frac{\partial Z}{\partial x_k} - v^k \frac{\partial Z}{\partial x_\ell}\right) d\Omega$$

$$= -\frac{e_0^2}{2a^2}\int \frac{\partial Z}{\partial x^\ell}\frac{\partial Z}{\partial x_\ell}\, d\Omega, \qquad (4.14)$$

since

$$v_k \frac{\partial Z}{\partial x_k} = v_k \frac{\partial \mu}{\partial x_k}\frac{\partial Z}{\partial \mu} = v_k m^{k\ell} m\gamma v_\ell \cdot \frac{1}{\mu}\frac{\partial Z}{\partial \mu} = 0 \qquad (4.15)$$

Hence, by Fourier transfor~~m~~ms,

$$L^e = -\frac{e_0^2}{2a^2\gamma^2}\iiint \frac{a^2\hbar^2}{(2\pi\hbar)^6} \frac{e^{i(\underline{k}-\underline{\ell})\cdot(\underline{x}-\underline{v}t)/\hbar}\, e^{\{(\underline{k}\cdot\underline{v})^2/c^2 - k^2\}/2b^2}\, e^{\{(\underline{\ell}\cdot\underline{v})^2/c^2 - \ell^2\}/2b^2}\, \{(\underline{k}\cdot\underline{v})(\underline{\ell}\cdot\underline{v})/c^2 - \underline{k}\cdot\underline{\ell}\}\, d\Omega\, d\underline{k}\, d\underline{\ell}}{\{k^2 - (\underline{k}\cdot\underline{v})^2/c^2\}\{\ell^2 - (\underline{\ell}\cdot\underline{v})^2/c^2\}}$$

$$= \frac{e_0^2\hbar^2}{2\gamma^2(2\pi\hbar)^3}\int \frac{e^{\{(\underline{k}\cdot\underline{v})^2/c^2 - k^2\}/b^2}\, d\underline{k}}{k^2 - (\underline{k}\cdot\underline{v})^2/c^2} = \frac{e_0^2\hbar^2}{2\gamma^2(2\pi\hbar)^3}\int \frac{e^{-(k_1^2/\gamma^2 + k_2^2 + k_3^2)/b^2}\, d\underline{k}}{k_1^2/\gamma^2 + k_2^2 + k_3^2} \qquad (4.16)$$

$$= \frac{e_0^2}{(4\pi)^{3/2} a\gamma} = \frac{m_0c^2}{\gamma}$$

This is the <u>negative</u> of the relativistic Lagrangian of classical mechanics for a free particle with mass m_0, and the momentum and energy derived from this form of the Lagrangian in the usual way are $-m_0\gamma\underline{v}$ and $-m_0\gamma c^2$. In fact the momentum and energy of the singularity, regarded as a particle, turn out to be equal and opposite to the momentum and energy of the associated field.

In this way a very special form of the conservation of momentum and energy appears to be satisfied for the singularity alone. As a corollary to this result, which is obviously a particular instance of a much more general theorem, it may be noted that one should take, for the Lagrangian of the singularity, $+L^e$ if this is regarded as a functional of the field variable and its derivatives, but $-L^e$ if it is regarded as a functional of the coordinates and velocity of the singularity.

5. The Problem of Electromagnetic Interaction.

It is now possible to approach the problem of the interaction between an electronic singularity of the type discussed in the previous section with the radiation field considered earlier. It has already been seen that the Lagrangian for the field, regarded as a functional of the vector potential A^f_k of the field and its derivatives, is

$$L^f = \frac{1}{2}\int \left\{ F^{kl}(p_k, p'_k) A^f_l(x_k) A^f_k(x'_k) \right\}_{x'_k = x_k} d\Omega \tag{5.1}$$

while the Lagrangian for the singularity, regarded as a functional of its position x^0_k and velocity v_k, is

$$-L^e = -\frac{1}{2}\int \left\{ F^{kl}(p_k, p'_k) A^e_l(x_k) A^{\cancel{f}}_k(x'_k) \right\}_{x'_k = x_k} d\Omega, \quad A^e(x_k) = \frac{e_0}{a} Z(x^k - x^k_0, v_k). \tag{5.2}$$

In constructing the term representing the interaction, one is guided by the desideratum that the field equation for A^f_k should be

$$F^{kl}(p_k, p_k) A^f_l(x_k) = \frac{e_0 v^k}{c}\delta(\underline{r}) = F^{kl}(p_k, p_k) A^e_l(x_k), \quad r^k = x^k - x^k_0(t) \tag{5.3}$$

This determines the Lagrangian of interaction unambiguously in the form

$$L^{ef} = -\frac{1}{2}\int \left[F^{kl}(p_k, p'_k) \left\{ A^e_l(x_k) A^f_k(x'_k) + A^f_l(x_k) A^e_k(x'_k) \right\} \right]_{x'_k = x_k} d\Omega . \tag{5.4}$$

It is satisfactory to note that the Lagrangian densities contained in all three integrals are of the form required by the Principle of Reciprocity. Variation of the total Lagrangian

$$L = -L^{e} + L^{ef} + L^{f} \tag{5.5}$$

with respect to the field variable A^{f}_{k} leads to (5.3), which has the solution

$$A^{f}_{k} = A^{e}_{k} + A^{r}_{k}, \tag{5.6}$$

where A^{r}_{k} is the vector potential of a pure radiation field, satisfying

$$F^{kl}(p_k, p_k)\, A^{r}_{l}(x_k) = 0 \tag{5.7}$$

It may be noted that the solution of the equation (5.3) for A^{e}_{k} is given exactly in (5.2) only when the motion of the charge is rectilinear; in general there may be corrections neglect of which in this section is equivalent to the neglect of the radiation damping of classical electrodynamics. These corrections will be determined in the following section.

To compare the Lagrangian (5.4) of interaction with the usual one, it is convenient to transform the integrand, with the help of (3.10), in the following way:

$$\begin{aligned}
\{F^{kl}(p_k, p_k')\, A^{e}_{l}(x_k)\, A^{f}_{k}(x_k')\} &= -\frac{\partial B^{e}_{l}}{\partial x^{k}}\frac{\partial B^{fl}}{\partial x_{k}} + \frac{\partial B^{e}_{k}}{\partial x^{l}}\frac{\partial B^{fl}}{\partial x_{k}} \\
&= \Box B^{e}_{k} B^{fk} + \frac{\partial}{\partial x_{l}}\left(E^{e}_{kl} B^{f}_{k}\right) \\
\{F^{kl}(p_k, p_k')\, A^{f}_{l}(x_k)\, A^{e}_{k}(x_k')\} &= B^{fk}\Box B^{e}_{k} + \frac{\partial}{\partial x_{l}}\left(B^{fk} E^{e}_{kl}\right.
\end{aligned} \tag{(5.8)}$$

Then (5.4) becomes

$$L^{ef} = -\frac{e_0 v^k}{c}\int B^f_k(x_k)\,\Box Z(x_k)\,d\Omega - \frac{\partial}{\partial t}\int B^{f\ell}(x_k)\,E^e_{\ell 4}(x_k)\,d\Omega \tag{5.9}$$

The second term in (5.9) can obviously give no contribution to the field equations, which are derived by minimizing the action, or time integral of the Lagrangian, with stationary boundary conditions; also all observable quantities can be related to the field equations; this term may therefore be discarded without error. Hence (5.9) reduces to

$$L^{ef} = -\frac{ev^k}{c}C^f_k(x^k_o, v_k),\quad C^f_k(x^k_o, v_k) = \int B^f_k(x_k)\,\frac{e^{-\tilde{r}^2/2a^2}}{(2\pi)^{3/2}a^3}\,d\Omega, \tag{5.10}$$

$$\tilde{\underline{r}} = \{\underline{v}\wedge(\underline{r}\wedge\underline{v}) + \gamma(\underline{r}\cdot\underline{v} - v^2 r_4/c)\underline{v}\}/v^2,\quad r^k = x^k - x^k_o$$

The exponential factor in the integrand is for practical purposes a δ-function, except for ultra-relativistic velocities of the electron, and radiation whose wave-length is comparable with the electronic radius a. As these possibilities can be discussed more conveniently in the sections devoted to the quantum theory, they will be set aside for the present, and the exact function $C^f_k(x^k_o, v_k)$ replaced by $A^f_k(\underline{x}_o)$. Then the first two terms of the Lagrangian reduce to

$$-L^e + L^{ef} = -m_o c^2/\gamma - e_o v^k A^f_k(x^k_o)/c \tag{5.11}$$

and the straightforward variation of these with regard to the position $\underline{x}_o$ and velocity $\underline{v}$ of the electron leads, in the usual way, to

$$\frac{d}{dt}\left(m_0 \gamma \underline{v} + \frac{e_0}{c}\underline{A}^f\right) = -e_0 \frac{\partial}{\partial \underline{x}_0}\left(A_4^f - \underline{v}\cdot\underline{A}^f/c\right). \qquad (5.12)$$

Substituting for A_k^f from (5.6), and remarking that at the point $\underline{x}_0$, A_k^e reduces to $\frac{2e_0\gamma v_k}{(4\pi)^{3/2}ac}$, while $\frac{\partial A_k^e}{\partial \underline{x}}$ vanishes, one obtains

$$\frac{d}{dt}\left(m\gamma\underline{v} + \frac{e_0}{c}\underline{A}^r\right) = -e_0\frac{\partial}{\partial \underline{x}_0}\left(A_4^r - \underline{v}\cdot\underline{A}^r/c\right), \qquad (5.13)$$

which is the equation of motion of the electron. This calculation justifies the statement in the previous section that m_0 is only a fraction of the rest-mass

$$m = \frac{3e_0^2}{(4\pi)^{3/2}ac^2}, \qquad (5.14)$$

two thirds of which is derived from the field.

In this section the classical treatment of the interaction problem has been followed as closely as possible, the only deviation consisting in the explicit recognition that the field energy, which diverges in the customary theory but here converges, contributes directly to the energy of the electron. It may be remarked as a matter of interest that the sum of the three terms of the Lagrangian is

$$\int\left\{\tfrac{1}{2}F(A^f - A^e)(A^{f'} \;\; A^{e'}) - FA^eA^{e'}\right\}d\Omega, \quad A^f - A^e = A^r, \qquad (5.15)$$

provided only that A^e be treated as a functional of the position and velocity of the singularity, and not as a field variable in its own right. If it were regarded as a field variable, the correct Lagrangian would be

$$L = L^e + L^{ef} + L^f = \frac{1}{2}\int F(A^f - A^e)(A^{f\prime} - A^{e\prime})\, d\Omega, \tag{5.16}$$

which is, effectively, the Lagrangian of a pure radiation field.

6. Accelerated Motion and Radiation Reaction

An alternative method of deriving the equation of motion of a point charge, reviewed by Eliezer (24) for ordinary electrodynamics, and applied by Schrödinger (15) to Born's non-linear electrodynamics, is to solve exactly the field equation

$$F^{kl}A_l = \frac{ev^k}{c}\delta(\mathbf{x}-\mathbf{x}_0), \quad v^k = (\dot{\mathbf{x}}_0, c) \tag{6.1}$$

and afterwards to calculate the momentum flux across a small sphere surrounding the singularity. For this latter purpose it is obviously more convenient to determine $B_k = e^{-p_k p^k/2b^2}A_k$ than A_k itself. By Fourier transforms one obtains without difficulty

$$\Box B_k = e^{-\frac{a^2}{2c^2}\frac{\partial^2}{\partial t^2}}\frac{e_0 v_k}{c}\frac{e^{-\mathbf{r}^2/2a^2}}{(2\pi)^{3/2}a^3}, \quad \mathbf{r} = \mathbf{x}-\mathbf{x}_0 \tag{6.2}$$

Now, for a sufficiently 'well-behaved' function $f(t)$, $e^{-\frac{a^2}{2c^2}\frac{\partial^2}{\partial t^2}}f(t)$ is the coefficient of the constant term in the expansion in powers of τ of

$$\sum_{n=0}^{\infty}\left(-\frac{a^2}{2c^2}\right)^n\frac{(2n)!}{n!}\frac{f(t+\tau)}{\tau^{2n}}$$

Also

$$\begin{gathered}\exp\left[-\{\mathbf{r}(t+\tau)\}^2/2a^2\right] = \exp\left(\sigma + \frac{\sigma_1\tau}{1!} + \frac{\sigma_2\tau^2}{2!} + \dots\right), \\ -2a^2\sigma = \mathbf{r}^2, \quad -a^2\sigma_1 = \mathbf{r}\cdot\dot{\mathbf{r}} = -\mathbf{r}\cdot\mathbf{v}, \\ -a^2\sigma_2 = v^2 - \mathbf{r}\cdot\dot{\mathbf{v}}, \quad -2a^2\sigma_n = \frac{d^n}{dt^n}(\mathbf{r}^2).\end{gathered} \tag{6.3}$$

Hence

$$e^{-\frac{a^2}{2c^2}\frac{\partial^2}{\partial t^2}}e^{\sigma} = e^{\sigma}\sum_{n=0}^{\infty}\left(-\frac{a^2}{2c^2}\right)^n\frac{(2n)!}{n!}\sum_{n_i}\cdots\sum_{(\Sigma i n_i = 2n)}\frac{\left(\frac{\sigma_1}{1!}\right)^{n_1}\left(\frac{\sigma_2}{2!}\right)^{n_2}\cdots}{n_1!\,n_2!\cdots} \tag{6.4}$$

This expression is exact; if terms involving σ_3 , σ_4 , etc. are neglected, the right-hand side of (6.4) reduces to

$$e^{\sigma}\sum_{n=0}^{\infty}\left(-\frac{a^2}{2c^2}\right)^n\frac{(2n)!}{n!}\sum_{m=0}^{n}\frac{\sigma_1^{2m}\left(\frac{1}{2}\sigma_2\right)^{n-m}}{(2m)!\,(n-m)!}=e^{\sigma}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left(-\frac{a^2}{2c^2}\right)^{m+k}\frac{(2m+2k)!\,\sigma_1^{2m}\left(\frac{1}{2}\sigma_2\right)^k}{(m+k)!\,(2m)!\,k!} \qquad (6.5)$$

since a summation with respect to n from 0 to ∞ and m from 0 to n is equivalent to a summation with respect to both m and $k=n-m$ from 0 to ∞ . Performing the summation with respect to k first, one obtains

$$e^{\sigma}\sum_{m=0}^{\infty}\left(-\frac{a^2}{2c^2}\right)^m\frac{(2m)!}{m!}\frac{\sigma_1^{2m}}{(2m)!}\sum_{k=0}^{\infty}\frac{(2m+1)(2m+3)\cdots(2m+2k-1)}{2\cdot 4\cdots 2k}\left(-\frac{a^2\sigma_2}{c^2}\right)^k$$

$$=e^{\sigma}\sum_{m=0}^{\infty}\left(-\frac{a^2}{2c^2}\right)\frac{\sigma_1^{2m}}{m!}\left(1+\frac{a^2\sigma_2}{c^2}\right)^{-(m+\frac{1}{2})} \qquad (6.6)$$

$$=\gamma\exp(\sigma-\gamma^2a^2\sigma_1^2/2c^2),$$

where γ now means $(1-v^2/c^2+\underline{r}\cdot\dot{\underline{v}}/c^2)^{-\frac{1}{2}}$.

Similarly

$$e^{-\frac{a^2}{2c^2}\frac{\partial^2}{\partial t^2}}\underline{v}\,e^{\sigma}=\gamma\underline{v}\exp\left(\sigma-\frac{\gamma^2a^2\sigma_1^2}{2c^2}\right)+\dot{\underline{v}}\,e^{\sigma}\sum_{n=0}^{\infty}\left(-\frac{a^2}{2c^2}\right)^n\frac{(2n)!}{n!}\sum_{m=1}^{n}\frac{\sigma_1^{2m-1}\left(\frac{1}{2}\sigma_2\right)^{n-m}}{(2m-1)!\,(n-m)!}$$

$$=\gamma(\underline{v}-\gamma^2a^2\sigma_1\dot{\underline{v}}/c^2)\exp(\sigma-\gamma^2a^2\sigma_1^2/2c^2), \qquad (6.7)$$

by neglect of terms involving $\ddot{\underline{v}}$ and higher derivatives.

Having evaluated the right-hand side of (6.2), the solution is now easily completed by the method of ordinary electrodynamics. One has

$$B_4(\underline{x})=\frac{e_0}{(2\pi)^{3/2}a^3}\int\gamma\exp\left[-\frac{1}{2a^2}\left\{r'^2+\gamma^2(\underline{r}'\cdot\underline{v})^2/c^2\right\}\right]d\underline{x}'/|\underline{x}-\underline{x}'|$$

$$\underline{B}(\underline{x})=\frac{e_0}{(2\pi)^{3/2}a^3c}\int\gamma(\underline{v}+\gamma^2\underline{r}'\cdot\underline{v}\,\dot{\underline{v}}/c^2)\exp\left[-\frac{1}{2a^2}\left\{r'^2+\gamma^2(\underline{r}'\cdot\underline{v})^2/c^2\right\}\right]d\underline{x}'/|\underline{x}-\underline{x}'| \qquad (6.8)$$

$$(\underline{r}'=\underline{x}'-\underline{x}_0),$$

which is the approximate solution for accelerated motion. If the usual assumptions are introduced that v and $\dot{v}$ are small, so that v^2/c^2 and $r\dot{v}/c^2$ may be neglected, the following simplification is effected:

$$B_k(\underline{x}) = \frac{e_0 v_k}{(2\pi)^{3/2} a^3 c} \int e^{-\underline{r}'^2/2a^2} d\underline{x}'/|\underline{x}-\underline{x}'| = \frac{e_0 v_k}{ac} Z(\tilde{r}) \tag{6.9}$$

where $r^k r_k = 0$, and $\tilde{r}$ is the retarded distance $r^k v_k/c$.

Now $\frac{\partial r^k}{\partial x^l} = \delta^k_l - v^k \frac{\partial t_0}{\partial x^l}$, and as $r_k \frac{\partial r^k}{\partial x^l} = 0$, $\frac{\partial t_0}{\partial x^l} = \frac{r_l}{r^k v_k}$; hence $\frac{\partial r^k}{\partial x^l} = \delta^k_l - \frac{v^k r_l}{r^m v_m}$ and $\frac{\partial v^k}{\partial x^l} = \dot{v}^k \frac{\partial t_0}{\partial x^l} = \frac{\dot{v}^k r_l}{r^m v_m}$. With the help of these relations, one has, to a sufficient approximation,

$$\frac{\partial B_k}{\partial x^l} = \frac{e_0}{a}\left\{ \frac{\dot{v}_k r_l}{c^2 \tilde{r}} Z(\tilde{r}) + \frac{v_k}{c} Z'(\tilde{r})\left(\frac{\dot{v}_m r^m r_l}{c^2 \tilde{r}} + \frac{v_l}{c} - \frac{r_l}{\tilde{r}} \right)\right\}$$

$$\frac{\partial B_l}{\partial x^k} - \frac{\partial B_k}{\partial x^l} = \frac{e_0}{a}\left\{ \frac{(\dot{v}_l r_k - \dot{v}_k r_l)}{c^2 \tilde{r}} Z(\tilde{r}) + \frac{(v_l r_k - v_k r_l)}{c \tilde{r}} Z'(\tilde{r})\left(\frac{r^m \dot{v}_m}{c^2} - 1 \right)\right\} \tag{6.10}$$

Hence

$$\underline{E} = -\left(\frac{\partial B_4}{\partial \underline{x}} + \frac{\partial \underline{B}}{\partial x_4} \right) = \frac{e_0}{\tilde{r} a}\left(1 + \underline{r}\cdot\dot{\underline{v}}/c^2\right) Z'(\tilde{r}) (\underline{r} - r\underline{v}/c) + \frac{e_0}{\tilde{r} a} Z(\tilde{r})\, r \dot{\underline{v}}/c^2$$

$$\underline{H} = \frac{\partial}{\partial \underline{x}} \wedge \underline{B} = -\frac{e_0}{\tilde{r} a}\left(1 + \underline{r}\cdot\dot{\underline{v}}/c^2\right) Z'(\tilde{r})\, \underline{r} \wedge \underline{v}/c + \frac{e_0}{\tilde{r} a} Z(\tilde{r})\, \underline{r} \wedge \dot{\underline{v}}/c^2 \tag{6.11}$$

For large r, $Z(\tilde{r}) = \frac{a}{4\pi \tilde{r}}$ and $Z'(r) = -\frac{a}{4\pi \tilde{r}^2}$, so that these field quantities are the same as those derived from Maxwell's electrodynamics; this ensures that the radiation field at a large distance from the accelerating electron is the same as predicted by the classical theory. As $r \to 0$, however,

$Z(\tilde{r}) \to \frac{1}{2}(2\pi)^{-3/2}$ and $Z'(\tilde{r}) \sim \frac{1}{6}\frac{(2\pi)^{-3/2}}{a^2}\tilde{r}$, so thatthe field quantities near the singularity have the finite values

$$\underline{E} = \frac{e_0}{2(2\pi)^{3/2}a}\,\frac{1}{1-\check{\underline{r}}\cdot\underline{v}/c}\,\frac{\dot{\underline{v}}}{c^2}\,,\quad \underline{H} = \check{\underline{r}}\wedge\underline{E}\,,\quad \check{\underline{r}} = \underline{r}/r \qquad (6.12)$$

This shows that there is no reaction on the singularity itself. If, however, a sphere of radius κa is described about the singularity, where κ is much greater than 1 but not too large to invalidate the approximation $\dot{\underline{v}}(t-\frac{\kappa a}{c}) = \dot{\underline{v}}(t) - \frac{\kappa a}{c}\ddot{\underline{v}}(t)$, one finds in the usual way the total energy flux $\frac{e_0^2\dot{v}^2}{6\pi c^3}$, and the momentum flux $\frac{e_0^2\dot{\underline{v}}}{6\pi\kappa a c^2} - \frac{e_0^2\ddot{\underline{v}}}{6\pi c^3}$, where the second term can be identified with the radiation reaction. The reaction is, in fact, a property of the field surrounding the singularity rather than the singularity itself.

7. The Quantum Theory of the Electron

The treatment of the electron has hitherto been of a quasi-classical character. The difficulty, peculiar to quantum mechanics, now has to be faced, how it is possible to reconcile the determinacy of the structure of the electronic field with the indeterminacy of its position required by Heisenberg's uncertainty principle. From the present unitary standpoint the practice of current quantum electrodynamics, which is to replace the point singularity by a continuous charge distribution proportional to $\psi^{\dagger}\alpha_4\psi$, where ψ is the solution of Dirac's equation, could not possibly be right, for this would lead to the absurd conclusion that the rest-energy of the electron depended upon the certainty with which its position was known. The error is easy to detect: it is not a fact that the charge of an electron whose momentum is exactly known is uniformly distributed over all space; on the contrary, its charge is as compact as ever, and only the <u>probability</u> of finding it located in a definite position is small and uniformly distributed. The only reason why this antinomy has not been emphasised before is that the divergence of the self-energy of the electron has made it a matter of little concern whether this value could be expected to depend on whether its position were known exactly or not. As soon as the energy is made finite, however, this becomes an outstanding problem. It is evaded in Podolsky's finite electrodynamics, and it is for this reason that it is there impossible to give an adequate representation of any definite physical situation.

It may be noted as a preliminary consideration that although the exact knowledge of both the position and momentum of an electron cannot, according to the uncertainty principle, be assumed, there is no difficulty in assigning a physical meaning to certain combinations of them, like the resultant angular momentum. It has already been seen that the solution of the equation

$$e^{a^2\Box}\Box A_k = e_0\,\delta(\tilde{\underline{r}})\frac{p_k^0}{mc}, \quad \underline{r} = \underline{n}\cdot\underline{p}^0\,\underline{p}^0/mc\,p^{02} + \underline{p}^0\wedge\underline{m}/p^{02},$$
$$\underline{m} = \underline{r}\wedge\underline{p}^0, \quad \underline{n} = \underline{r}\,p_4^0 - r_4\,\underline{p}^0, \quad mc = (p_4^{02} - \underline{p}^{02})^{\frac{1}{2}}, \quad r^k = x^k - x_0^k \qquad (7.1)$$

can be expressed in the form

$$A_k = \frac{e_0}{(2\pi)^{3/2}\sqrt{2a}}\,Y\!\left(\frac{\mu}{mca}\right)\frac{p_k^0}{mc}, \qquad \mu^2 = \underline{n}^2 - \underline{m}^2,$$

and there is little difficulty in interpreting this solution in the quantum theory. When the scalar angular momentum μ is not diagonal, one writes

$$A_k \phi(x_0^k) = \frac{e_0}{2\pi\hbar^2\sqrt{2a}} Y\left(\frac{\mu}{mca}\right)\frac{p_k^0}{mc}\phi(x_0^k) \tag{7.2}$$

where $\phi(x_0^k)$ contains the wave amplitude for the electron; for comparison with the orthodox theory it may be expressed in theform

$$\phi(x_0^k) = \Omega^{-1}\sum_{p_k^0}\frac{1}{2mc}(1+\alpha^k p_k^0)e^{-ix_0^k p_k^0/\hbar}\psi(p_k^0) \tag{7.3}$$

so that $\alpha^k p_k^0 \phi = mc\phi$, and, on multiplying (7.2) by α^k , one obtains

$$A\phi = \alpha^k A_k \phi = \frac{e_0 Y\phi}{(2\pi)^{3/2}\sqrt{2a}} \tag{7.4}$$

All difficulties connected with non-commuting quantities will now disappear provided the right-hand side of (7.4) is interpreted to mean

$$Y\phi(x_0^k) = \Omega^{-1}\sum_{p_k^0} Y(x^k - x_0^k; p_k^0)\frac{1}{2mc}(1+\alpha^k p_k^0)e^{-ix_0^k p_k^0/\hbar}\psi(p_k^0) \tag{7.5}$$

It is now obvious from the corresponding classical calculation that the equation

$$e^{a^2\Box}\Box Y\phi - \hbar^2\delta(r)\phi \tag{7.6}$$

is satisfied, provided that the right-hand side is interpreted in a similar way.

The quantum theory of the electron may therefore be developed from the equation

$$e^{a^2\Box}\Box\Phi = e_0\delta(\underline{r})\phi, \quad \Phi = A(x_k)\phi(x_0^k), \tag{7.7}$$

where Φ is regarded as a composite wave vector for the system of field and singularities together.

The corresponding statistical operator is

$$\rho(x^k, x_0^k; x^{k\prime}, x_0^{k\prime}) = A(x_k)\rho_0(x_0^k, x_0^{k\prime})A^*(x_k) = A(x^k)\phi(x_0^k)\phi^*(x_0^{k\prime})A^*(x^{k\prime}). \tag{7.8}$$

The result is a field theory similar to that developed in the second half of § 3. As can be seen from the corresponding calculation in the classical theory, the Lagrangian for the electron is

$$L^e = \int sp\{\eta e^{-\frac{1}{2}\eta^2/b^2}\rho\,\eta' e^{-\frac{1}{2}\eta'^2/b^2}\}_{x'_k=x_k} d\Omega$$
$$= 2m_0c^2\phi^*(x_0^k)\frac{mc}{p_4}\phi(x_0^k) \qquad (7.9)$$

Now, from (7.3) one has

$$\frac{1}{2mc}(mc+\alpha^k p_k^0)\phi = \phi$$
$$\phi^*\alpha_4\frac{1}{2mc}(mc+\alpha^k p_k^0)\alpha_4 = \phi^* \qquad (7.10)$$

taking account of the fact that $\underline{\alpha}$ is anti-hermitian.

Hence

$$L^e = 2\frac{m_0}{m}\phi^*\frac{\alpha_4(mc+\alpha^k p_k^0)}{2p_4^0}\cdot cp_4^0\cdot\frac{\alpha_4(mc+\alpha^k p_k^0)}{2p_4^0}\phi$$
$$= \frac{m_0}{m}\phi^*\alpha_4(mc+\alpha^k p_k^0)c\phi \qquad (7.11)$$

using the fact that $\frac{\alpha_4}{2p_4^0}(mc+\alpha^k p_k^0)$ is an idempotent operator. This is the form of the Lagrangian of an electron field recognised in current quantum theory, though again the sign is strictly incorrect. It is well known that the Lagrangian $-L^e$ provides the field equation

$$(mc+\alpha^k p_k^0)\phi = 0 \qquad (7.12)$$

of the electron, and that the energy density is

$$\mathcal{H}^e = \frac{m_0}{m}\phi^*\alpha_4(\underline{\alpha}\cdot\underline{p}+mc)\phi \qquad (7.13)$$

One has, therefore, in $-L^e$, a satisfactory Lagrangian for the electron regarded as a <u>particle</u>.

From this starting point, the theory of the electron can clearly be developed in the usual way. The novelty in the present treatment consists in showing that the same results can be derived from a Lagrangian of the same form as required for the radiation field, and that the electron can still in quantum theory be regarded as a singularity in the photon field.

8. The Quantum Theory of Electromagnetic Interaction

The preceding theory of the electron will now be extended to meet the physical situation where it is in interaction with the photon field, on lines closely similar to the classical theory of § 5. The field is treated in the way suggested by the second half of § 3, with a Lagrangian

$$L_f = \int sp\{F(\eta,\eta)\Phi_f\Phi_f^*\}_{x_k'=x_k} d\Omega \quad , \quad \Phi_f = A_f\phi \quad , \quad \Phi_f^* = \phi^* A_f^* \tag{8.1}$$

As seen in the previous section, the Lagrangian of the electron, regarded as a particle, is

$$\begin{aligned} -L_e &= -\int sp\{F(\eta,\eta)\Phi_e\Phi_e^*\}_{x_k'=x_k} d\Omega \\ &= -\frac{m_0}{m}\phi^\dagger(\alpha^k p_k^0 + mc)\phi \ . \end{aligned} \tag{8.2}$$

To obtain the desired field equation

$$F(\eta,\eta)\Phi_f = e_0\delta(\tilde{x})\phi = F(\eta,\eta)\Phi_e \tag{8.3}$$

it is necessary to choose as the Lagrangian of interaction

$$\begin{aligned} L_{ef} &= -\int sp\{F(\eta,\eta')(\Phi_e\Phi_f^* + \Phi_f\Phi_e^*)\}_{x_k'=x_k} d\Omega \\ &= -\int sp\{F(\eta,\eta)(A_e\phi\phi^*A_f^* + A_f\phi\phi^*A_e^*)\}_{x_k'=x_k} d\Omega \end{aligned} \tag{8.4}$$

This is to be regarded as a functional of ϕ , ϕ^* , A_f , and A_f^* ; just as in the classical instance, it can be evaluated for comparison with the orthodox theory in the following way. If $B_f = e^{-\eta^2/2b^2}A_f$, one has

$$\begin{aligned} \int sp\{F(\eta,\eta')A_e\phi\phi^*A_f^*\}d\Omega &= \frac{e_0}{a}\int\{\phi^* B_f^*(x_k')\eta'\eta\, Z(x_k)\phi\}d\Omega \\ &= \frac{e_0}{a}\int\{\phi^* B_f^*(x_k')\{\eta^2 + (\eta'-\eta)\eta\}Z(x_k)\phi\}d\Omega \\ &= \frac{e_0}{a}\int\phi^* B_f^*(x_k)\{\eta^2 Z(x_k)\}\phi\, d\Omega - \frac{i\hbar e_0}{a}\frac{\partial}{\partial t}\int\phi^* B_f^*(x_k)\alpha_4\{\eta Z(x_k)\}\phi\, d\Omega \end{aligned} \tag{8.5}$$

where the second term does not contribute to the field equations (which are derived from the <u>action</u>), and may be discarded. Hence the effective Lagrangian for the electron may be written in the form

$$-L_e+L_{ef}=-\frac{m_0c}{m}\phi^*\alpha_4(\alpha^k p_k^0+mc)\phi-\frac{e_0}{a}\int\phi^*[B_f^*(x_k)\{\eta^2 Z(x_k)\}+\{\eta^2 Z^*(x_k)\}B_f(x_k)]\,\phi\,d\Omega$$

$$=-\frac{m_0c}{m}\phi^\dagger(\alpha^k p_k^0+mc)\phi-\frac{e_0}{a}\int\phi^\dagger[B_f^\dagger(x_k)\{\eta^2\gamma Z(x_k)\}+\{\eta^2\gamma Z^*(x_k)\}B_f(x_k)]\,\phi\,d\Omega, \quad (8.6)$$

$$\phi^\dagger=\phi^*\alpha_4\,,\quad B_f^\dagger=\alpha_4 B_f^*\alpha_4\,,\quad \gamma=p_4^0/mc\,.$$

The equation of motion, derived by variation with respect to $\phi^\dagger$, is

$$\frac{m_0c}{m}(\alpha^k p_k^0+mc)\phi+\frac{e_0}{a}\int[B_f^\dagger(x_k)\{\eta^2\gamma Z(x_k)\}+\{\eta^2\gamma Z^*(x_k)\}B_f(x_k)]\,\phi\,d\Omega=0\,. \quad (8.7)$$

This is the analogue of Dirac's equation for the electron in interaction with the electromagnetic field. The 'longitudinal field' can now be eliminated by the substitution of the solution

$$A_f\phi=A_e\phi+A_r\phi\,,\quad B_f\phi=B_e\phi+B_r\phi\,,$$
$$B_r\phi=e^{-\eta^2/2b^2}A_r\phi=A_r\phi\,,\quad B_e\phi=e^{-\eta^2/2b^2}A_e\phi=\frac{e_0}{a}Z\phi \quad (8.8)$$

of the field equation (8.3), where $\Phi_r=A_r\phi$ is the solution of

$$F(\eta,\eta)\,\Phi_r=0\,. \quad (8.9)$$

The terms involving A_e alone are simplified by the already familiar evaluation of the space integral, and (8.7) then reduces to

$$\frac{3m_0c}{m}(\alpha_k p_k^0+mc)\phi+\frac{e_0}{a}\int[B_r^\dagger(x_k)\{\eta^2\gamma Z(x_k)\}+\{\eta^2\gamma Z^*(x_k)\}B_r(x_k)]\,\phi\,d\Omega=0\,. \quad (8.10)$$

As in the classical theory, the observed rest-mass is seen to be about three times the value which arises from the field of the singularity alone; even the factor 3 is not quite correct owing to a much smaller contribution from the 'transverse field'.

To complete the comparison with the usual theory, it is necessary to evaluate the integral contained in (8.10); this is effected by introducing the Fourier transformation

$$B_r(x^k) = \frac{1}{(2\pi\hbar)^3}\int \alpha^l B_l(\underline{k})\, e^{i(\underline{k}\cdot\underline{x}-ckt)/\hbar}\, d\underline{k} \tag{8.11}$$

and making use of the Fourier transform of given by (4.13). Then

$$\int B_r^{\dagger}(x_k)\{\eta^2\gamma Z(x_k)\}\phi\, d\Omega = \frac{e_0}{(2\pi\hbar)^6}\iiint \alpha^m B_m^*(\underline{k})\, e^{-i(\underline{k}\cdot\underline{x}-ckt)/\hbar}\, e^{\{(\underline{l}\cdot\underline{v})^2/c^2-l^2\}/2b^2}\, e^{i\underline{l}\cdot\{\underline{x}-\underline{x}_0-\underline{v}(t-t_0)\}/\hbar}\phi\, d\Omega\, d\underline{k}\, d\underline{l}$$

$$= \frac{e_0}{(2\pi\hbar)^3}\int \alpha^m B_m^*(\underline{k})\, e^{-i(\underline{k}\cdot\underline{x}_0-ckt_0)/\hbar}\, e^{\{(\underline{k}\cdot\underline{v})^2/c^2-k^2\}/2b^2}\phi\, d\underline{k}\,, \quad (t_0=t) \tag{8.12}$$

where $\underline{v} = c\,p_0/(p_0^2+m^2c^2)^{\frac{1}{2}}$, and the momentum operator p_0 acts on ϕ. Similarly

$$\int\{\eta^2\gamma Z^*(x_k)\}B_r(x_k)\phi\, d\Omega = \frac{e_0}{(2\pi\hbar)^3}\int e^{\{(\underline{k}\cdot\underline{v})^2/c^2-k^2\}/2b^2}\alpha^m B_m(\underline{k})\, e^{i(\underline{k}\cdot\underline{x}_0-ckt_0)/\hbar}\phi\, d\Omega, \tag{8.13}$$

where p_0 now acts on $e^{i\underline{k}\cdot\underline{x}_0/\hbar}\phi$, instead of on ϕ alone. The modified form of Dirac's equation is therefore

$$\frac{3m_0c}{m}(\alpha^k p_k^0+mc)\phi + e_0\alpha^m\int\left[B_m^*(\underline{k})\, e^{-i\underline{k}\cdot\underline{x}_0/\hbar}\, e^{\{(\underline{k}\cdot p_0)^2/E_0^2-k^2\}/2b^2} B_m(\underline{k})\, e^{i\underline{k}\cdot\underline{x}_0/\hbar}\right]\phi\, d\underline{k} \tag{8.14}$$

where E_0 is the energy $(m^2c^2+p_0^2)^{\frac{1}{2}}$.

From this it is evident that the customary theory is modified only by the presence of a factor $e^{\{(\underline{k}\cdot p_0)^2/E_0^2 - k^2\}/2b^2}$ in the matrix element representing the absorption or emission of a photon with momentum $\underline{k}$ by the electron. Here (p_0, E_0) is to be interpreted as the energy momentum four-vector of the electron in the initial state when the photon is emitted, in the final state when the photon is absorbed. This is confirmed by the Hamiltonian formulation, which is the usual point of departure in quantum electrodynamics. The equation (8.11) is transformed into Hamiltonian form simply by multiplication by $\frac{m\alpha_4}{3m_0}$. This shows that the Hamiltonian energy operator is

$$H = \frac{mc\alpha_4}{3m_0}(\underline{\alpha}\cdot p_0 + mc) + \frac{me_0\alpha_4\,\underline{\alpha}\cdot}{3m_0(2\pi\hbar)^3}\int\Big[\underline{B}^*(\underline{k})\,e^{-i\underline{k}\cdot\underline{x}_0/\hbar}\,e^{\{(\underline{k}\cdot p_0)^2/E_0^2 - k^2\}/2b^2} + e^{\{(\underline{k}\cdot p_0)^2/E_0^2 - k^2\}/2b^2}\,\underline{B}(\underline{k})\,e^{i\underline{k}\cdot\underline{x}_0/\hbar}\Big]\,d\underline{k} + H_r \qquad (8.15)$$

where H_r is the energy of the pure radiation field, which is

$$H_r = \frac{1}{\hbar^2(2\pi\hbar)^3}\int \underline{B}^*(\underline{k})\cdot\underline{B}(\underline{k})\,k^2\,d\underline{k} \qquad (8.16)$$

according to (3.14): $\underline{B}$ is of course the same as $\underline{A}$ for this part of the field. The commutation rules (3.15) determine the matrix elements of $\underline{B}$ when the number of photons in any range of momentum is diagonal, which are the same as in the usual theory.

Thus the effect of the substitution of the reciprocally invariant Lagrangian operator for the usual one in the theory of the interaction of electrons and photons is summarized in the following simple prescription: to the matrix elements of the perturbation energy representing the emission of a photon with momentum $\underline{k}$ by an electron with momentum $\underline{p}_0$, or the absorption of a photon with momentum $\underline{k}$ by an electron with momentum $\underline{p}_0-\underline{k}$, join a factor $e^{\{(\underline{k}\cdot\underline{p}_0)^2/(p_0^2+m^2c^2)-k^2\}/2b^2}$. Some of the consequences of this rule will be traced in the following section.

9. Elementary Consequences

The most immediate application of the rule derived in the previous section is to the calculation of the transverse self-energy of the electron, which may be regarded as due to the emission and instantaneous reabsorption of photons with an infinite range of momenta. A typical process is represented schematically by

$$(1)\ \underline{p} \xrightarrow{+\underline{k}} (2)\ \underline{p}+\underline{k} \xrightarrow{-\underline{k}} (3)\ \underline{p}$$

which shows the successive values of the momentum of the electron. Ignoring for the moment the effect of the presence of the electron in suppressing fluctuations of the vacuum, the well-known formula for the transverse energy is

$$\delta mc^2 = \frac{e_0^2}{4\pi^2\hbar cm}\int k\,dk$$

for $\underline{p}=0$. The integrand is essentially a product of the matrix elements for the transitions (1) to (2) and (2) to (3), and so, according to the rule, ought to contain a factor $e^{-k^2/2b^2}\cdot e^{-k^2/2b^2}$. Then one has the finite but rather large value

$$\delta mc^2 = \frac{e_0^2}{4\pi\hbar cm}\int_0^\infty e^{-k^2/b^2}k\,dk = \frac{e_0^2 b^2}{8\pi^2\hbar cm} \simeq \frac{137}{2\pi}mc^2. \qquad (b \simeq 4\pi\hbar c\cdot mc/e_0^2) \tag{9.1}$$

The situation is, however, different if one uses Weisskopf's value (25)

$$\delta mc^2 = \frac{e_0^2 mc}{4\pi\hbar}\int\frac{k\,dk}{m^2c^2+k^2+k(m^2c^2+k^2)^{\frac{1}{2}}}, \tag{9.2}$$

which takes account of the distribution of negative energy electrons predicted by the theory of holes. In applying the rule to Weisskopf's calculation, it has to be remembered that a positive electron is a <u>hole</u>, so that the virtual <u>absorption</u> of a photon by an electron filling a negative energy state has to be interpreted as the <u>emission</u> of a photon by a positron, and conversely. Then

$$\delta mc^2 \simeq \frac{e_0^2 mc}{8\pi\hbar}\int_{mc}^{\infty}\frac{e^{-k^2/b^2}dk}{k} \simeq \frac{2mc^2}{137} \qquad (9.2)$$

which is quite small. The transverse self-energy is therefore 'cut off' to quite a reasonable value by the damping factor.

Next the Klein-Nishina formula for the scattering cross-section of photons with an electron will be considered. Here it might be feared that the damping factor would reduce the scattering of quanta with high energy to negligible proportions, thus contradicting the experimental finding that the formula holds, at least as regards order of magnitude, up to the highest energies known, compared with which even bc is small. It will be found, however, that this is not so: only the purely virtual processes are cut off at high energies. The effective processes in Compton scattering are represented by

$$A: \quad (1)\ \underline{p} \xrightarrow{+\underline{k}_0} (2)\ \underline{p}+\underline{k}_0 \xrightarrow{-\underline{k}} (3)\ \underline{p}+\underline{k}_0-\underline{k}$$

$$B: \quad (1)\ \underline{p} \xrightarrow{-\underline{k}} (2)\ \underline{p}-\underline{k} \xrightarrow{+\underline{k}_0} (3)\ \underline{p}+\underline{k}_0-\underline{k}$$

and the Klein-Nishina formula

$$d\phi = \left(\frac{e_0^2 k}{4\pi mc^2 k_0}\right)^2 \frac{S}{8mc} \cdot 2\pi \sin\theta \, d\theta, \qquad \underline{k}\cdot\underline{k}_0 = k k_0 \cos\theta, \tag{9.3}$$

for the differential cross-section of a scattering process in which the initial and final directions of polarization $\underline{n}_0$ and $\underline{n}$ contain the angle Θ, consists of three terms, given by [c.f. Heitler (26) pp. 146-54]

$$\begin{aligned} S &= S_A - 2S_{AB} + S_B, \\ S_A &= 2E + \tfrac{1}{2}\,\mathrm{sp}\,(\alpha\,\underline{\alpha}\cdot\underline{p}\,\alpha\,\underline{\alpha}\cdot\check{\underline{k}}), \\ S_{AB} &= (E+mc) + (E-mc)\,\check{\underline{k}}\cdot\check{\underline{k}}_0 - 4\cos^2\Theta\, mc + \tfrac{1}{4}\,\mathrm{sp}\,(\alpha_0\,\underline{\alpha}\cdot\underline{p}\,\alpha_0\,\underline{\alpha}\cdot\check{\underline{k}} + \alpha\,\underline{\alpha}\cdot\underline{p}\,\alpha\,\underline{\alpha}\cdot\check{\underline{k}}_0), \\ S_B &= 2E + \tfrac{1}{2}\,\mathrm{sp}\,(\alpha_0\,\underline{\alpha}\cdot\underline{p}\,\alpha_0\,\underline{\alpha}\cdot\check{\underline{k}}_0), \\ \alpha &= \underline{n}\cdot\underline{\alpha}, \quad \alpha_0 = \underline{n}_0\cdot\underline{\alpha}, \quad \check{\underline{k}} = \underline{k}/k, \quad \check{\underline{k}}_0 = \underline{k}_0/k_0 . \end{aligned} \tag{9.4}$$

According to the rule, the correct formula is

$$d\phi = \frac{1}{8mc}\left(\frac{e_0^2 k}{4\pi mc^2 k_0}\right)^2 (f_A^2 S_A - 2 f_A f_B S_{AB} + f_B^2 S_B)\cdot 2\pi \sin\theta \, d\theta, \tag{9.5}$$

where

$$\begin{aligned} f_A &= \exp\left[k_0^4/(k_0^2+m^2c^2) - k_0^2 + (\underline{k}\cdot\underline{k}_0)^2/(k_0^2+m^2c^2) - k^2\right]/2b^2 \\ f_B &= \exp\left[-k^2 + \{\underline{k}\cdot(\underline{k}-\underline{k}_0)\}^2/\{m^2c^2+(\underline{k}-\underline{k}_0)^2\} - k_0^2\right]/2b^2 \end{aligned} \tag{9.6}$$

Now $k_0^2 - k_0^4/(k_0^2+m^2c^2) = k_0^2 m^2c^2/(k_0^2+m^2c^2)$ is always less than m^2c^2 ; and, according to the Compton formula $kk_0 - \underline{k}\cdot\underline{k}_0 = mc(k_0-k)$, one has

$$k^2 - (\underline{k}\cdot\underline{k}_0)^2/(k_0^2+m^2c^2) = \frac{k_0^2\sin^2\theta + m^2c^2}{\{mc + k_0(1-\cos\theta)\}^2} \qquad \frac{k_0^2 m^2c^2}{k_0^2+m^2c^2} \tag{9.7}$$

which is of the same order of magnitude except for large k_0 and $\theta \sim (2mc/k_0)^{\frac{1}{2}}$, when it reduces to mck_0. Then for $k_0 > (137)^2 mc$ and $\theta \sim 1/137$ there will be an appreciable deviation from $f_A = 1$, but elsewhere the correction is unobservable. In a similar way it can be seen that f_B differs appreciably from 1 only for quanta with energies near the limit of observation and scattering angles of the order of one half of one degree.

It is certain that discrepancies of this order cannot yet have been detected experimentally, though there is some hope that improvements in the technique of cosmic rays may make this possible in the future. A more rewarding direction in which to look for experimental confirmation of the theory may be in the small changes of the already small self-energy of the electron which are supposed to be responsible for the Lamb shift and similar phenomena. A detailed treatment of these questions would go rather beyond the scope of the present introductory paper, but it can be seen from (9.2) that the <u>order of magnitude</u> is right. Meanwhile it may be observed that although a qualitative justification of the subtraction theories is provided by the circumstance that the damping factor operates drastically only on those virtual processes involving what Heitler calls 'round-about' transitions, there are in fact small corrections to the results obtained by such methods.

10. Appendix I: The Symmetrical Energy-Momentum Tensor, and the Angular Momentum Tensor.

For the purpose of this section it is necessary to assume that the Lagrangian operator defined in (2.7) is Lorentz-invariant. Suppose then that the coordinates undergo an infinitesimal Lorentz transformation

$$x_k \longrightarrow x_k + \omega_{kl} x^l \tag{10.1}$$

so that the square of the anti-symmetrical tensor ω_{kl} is negligible, and that the statistical operator undergoes a corresponding transformation

$$\rho_{\alpha\beta} \longrightarrow \rho_{\alpha\beta} + \omega_{kl}\left(S^{kl}_{\alpha\gamma}\rho_{\gamma\beta} - \rho_{\alpha\gamma}S^{kl}_{\gamma\beta}\right). \tag{10.2}$$

It is useful to note that if the spin affixes α, β indicate transformation properties of the type associated with Dirac's anticommuting matrices α_k, then $S^{kl}_{\alpha\beta} = \frac{1}{8}\left(\alpha^k\alpha^l - \alpha^l\alpha^k\right)_{\alpha\beta}$; and if they have vectorial transformation properties, then $S^{\alpha\ kl}_{\cdot\beta} = \frac{1}{2}\left(g^{\alpha k}\delta^l_\beta - g^{\alpha l}\delta^k_\beta\right)$.

The wave operator F will undergo the transformation

$$F \longrightarrow F + \omega_{kl}\left(\frac{\partial F}{\partial p_k} p^l + \frac{\partial F}{\partial p'_k} p^{l\prime}\right) \tag{10.3}$$

hence, if $\mathcal{G}^{kl}(p_k, p'_k)$ and $\mathcal{G}^{kl\prime}(p_k, p'_k)$ are defined by

$$\left.\begin{aligned} \mathcal{G}^k(p_k, p'_k) &= \frac{\partial F(p_k, p'_k)}{\partial p_k} + (p'_l - p_l)\,\mathcal{G}^{kl}(p_k, p'_k) \\ \mathcal{G}^{k\prime}(p_k, p'_k) &= \frac{\partial F(p_k, p'_k)}{\partial p_k} + (p_l \quad p'_l)\,\mathcal{G}^{kl\prime}(p_k, p'_k) \end{aligned}\right\} \tag{10.4}$$

the Lagrangian operator will change by the quantity

$$sp \cdot \omega_{kl} \left[F(S^{kl}\rho - \rho S^{kl}) + (p_m' - p_m)(\mathcal{G}^{lm}p^k - \mathcal{G}^{lm'}p^{k'})\rho - (\mathcal{G}^l p^k + \mathcal{G}^{l'}p^{k'})\rho \right]$$

which, according to the inital assumption, must vanish. Next, the spin angular momentum is defined by

$$(i\hbar)^{-1}\mathcal{L}^{kl,m} = sp \cdot \left[\tfrac{1}{2}(\mathcal{G}^m S^{kl}\rho + \mathcal{G}^{m'}\rho S^{kl}) + \tfrac{1}{4}(\mathcal{G}^{km}p^l - \mathcal{G}^{lm}p^k - \mathcal{G}^{km'}p^{l'} + \mathcal{G}^{lm'}p^{k'})\rho \right] \tag{10.5}$$

Then it follows from the above that

$$(i\hbar)^{-1}(p_m' - p_m)\mathcal{L}^{kl,m} = sp\left\{ \tfrac{1}{4}(\mathcal{G}^k p^l - \mathcal{G}^l p^k + \mathcal{G}^{k'}p^{l'} - \mathcal{G}^{l'}p^{k'})\rho \right\} = \tfrac{1}{2}(\mathcal{L}^{kl} - \mathcal{L}^{lk}) \tag{10.6}$$

Hence the total angular momentum tensor

$$M^{kl,m} = \left\{ \mathcal{L}^{kl,m} + \tfrac{1}{2}(x^k \mathcal{L}^{ml} - x^l \mathcal{L}^{mk}) \right\}(x_k, x_k) \tag{10.7}$$

satisfies the conservation equation $\frac{\partial M^{kl,m}}{\partial x^m} = 0$; and a similar conservation equation is satisfied by the symmetrical energy-momentum tensor

$$T^{kl} = \left\{ \tfrac{1}{2}(\mathcal{L}^{kl} + \mathcal{L}^{lk}) + (i\hbar)^{-1}(p_m' - p_m)(\mathcal{L}^{mk,l} + \mathcal{L}^{ml,k}) \right\}(x_k, x_k) \tag{10.8}$$

11. Appendix II: A Theory of the Proton

In this section a modification of the quantized treatment of the electron will be given which seems to indicate the applicability of the theory to the nucleons. The equation of the electronic field, according to (7.7) and (7.10), is of the form

$$e^{a^2\Box}\Box\Phi = e_0\,\delta(\bar{r})\cdot\frac{1}{2mc}(\alpha^k p_k + mc)\psi,\qquad \frac{1}{2mc}(\alpha^k p_k + mc)\psi = \phi \tag{11.1}$$

The factors $\delta(\bar{r})$ and $\frac{1}{2mc}(\alpha^k p_k + mc)$ are actually non-commutative, but it will be found instructive to examine the consequences of reversing the order of these factors, so that (11.1) becomes

$$e^{a^2\Box}\Box\Phi = e_0\frac{1}{2mc}(\alpha^k p_k + mc)\,\delta(\bar{r})\psi = e_0\left\{\delta(\bar{r}) + \frac{i\hbar\alpha^k}{2mc}\frac{\partial}{\partial x^k}\delta(\bar{r})\right\}\psi \tag{11.2}$$

The singularity is not now radially symmetrical, but has a structure which allows the particle to have an angular motion whose contribution to the self-energy will be used to explain the large observed mass of the proton. The energy is essentially magnetic in origin, as is evident from the form of the co-factor $\frac{e_0\hbar}{2mc}\alpha^k m_{kl}p^l$ of the singularity. The following is, indeed, a refinement of a calculation by Born (18) purporting to show that the rest-mass of the proton is of a magnetic character.

The solution of the equation (11.2) is obviously

$$\begin{aligned}\Phi &= \tfrac{1}{2}(\alpha^k p_k + mc)\frac{e_0}{a}Y\left(\frac{\mu}{\hbar c}\right)\psi \\ &= \frac{e_0}{a}\left\{Y\left(\frac{\mu}{\hbar c}\right) + \frac{i\hbar\alpha_k}{2mc}\frac{\partial}{\partial x^k}Y\left(\frac{\mu}{\hbar c}\right)\right\}.\end{aligned} \tag{11.3}$$

From this, the energy of the field is readily calculated; it is, for the proton at rest,

$$\begin{aligned}M_0c^2 &= \frac{1}{2}\frac{e_0^2}{a}\int sp\left\{\delta(\underline{x}) + \frac{i\hbar\alpha^k}{2mc}\frac{\partial\delta(\underline{x})}{\partial x^k}\right\}\left\{Y(x) - \frac{i\hbar\alpha^k}{2mc}\frac{\partial Y(x)}{\partial x^k}\right\}d\Omega \\ &= \frac{1}{2}\frac{e_0^2}{a}\int \delta(\underline{x})\left(1 - \frac{\hbar^2}{4m^2c^2}\frac{\partial^2}{\partial\underline{x}\cdot\partial\underline{x}}\right)Y(\underline{x})\,d\Omega \\ &= \frac{e_0^2}{2a}\left\{Y(0) - \frac{\hbar^2}{4m^2c^2}\Delta Y(0)\right\} = \frac{e_0^2}{(4\pi)^{3/2}a}\left(1 + \frac{\hbar^2}{8m^2c^2a^2}\right).\end{aligned} \tag{11.4}$$

Just as for the electron, this does not represent the whole of the rest-energy Mc^2 of the proton in interaction with the electromagnetic field, but it is obvious that the respective masses are increased by the same factor 3. Hence

$$\frac{M}{m} = \frac{3e_0^2}{(4\pi)^{3/2}mc}\left(1 + \frac{\hbar^2}{8m^2c^2a^2}\right), \tag{11.5}$$

where the second term is much larger than the first. Assuming

$$m = \frac{\lambda e_0^2}{4\pi ac}, \tag{11.6}$$

one has

$$\frac{M}{m} = \frac{3}{2\lambda\sqrt{\pi}}\left\{1 + \frac{(137)^2}{8\lambda^2}\right\} = 1846, \tag{11.7}$$

provided $\lambda = 1{\cdot}025$. This value of λ agrees well with the value $\lambda = 1$ which was found necessary for the explanation of the mesonic masses, though, as has been seen, the value which can be inferred from the calculations of the present paper is not much greater than $0{\cdot}85$. For the explanation of this discrepancy, which, although not very large by itself, would considerably affect the calculated values of the masses of the mesons and proton, one has obviously to turn to phenomena which are not of an electromagnetic character. It is clear that interactions with other fields could make an appreciable difference to the mass of the light electron, while leaving the heavy proton almost unaffected. However this may be, the point of interest is that it is not impossible to reconcile the large mass of the proton with a particle of the same intrinsic nature as the electron.

At first sight the above theory might appear to lead to a magnetic coupling between the nucleons too large for comparison with experience; this is not so, however, as the expectation value of $\alpha_4^{(1)}\alpha_4^{(2)}$, which is the product of the α_4's for two different particles, is vanishingly small compared with $\alpha_4^2 = 1$, which determines the self-interaction. This and other questions remain to be investigated in detail.

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